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Nonnegative Matrices Having Same Nonnegative Moore–Penrose and Group Inverses

S. K. JAIN,*† V. K. GOEL‡ and EDWARD K. KWAK§

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Nonnegative matrices A whose Moore–Penrose generalized inverse A^+ is nonnegative and has any one of the three equivalent properties (i) $AA^+ = A^+A$, (ii) $A^+ = A^{\#}$, the group inverse, (iii) $A^+ = p(A)$, some polynomial in A with scalar coefficients, are characterized. This characterization generalizes known results on nonnegative matrices A whose Moore–Penrose generalized inverse is equal to some power of A.

1. INTRODUCTION

Let A be an $m \times n$ real matrix. Consider the Penrose [8] equations

$$AXA = A \tag{1}$$

$$XAX = X \tag{2}$$

$$(AX)^T = AX \tag{3}$$

$$(XA)^T = XA \tag{4}$$

where X is an $n \times m$ real matrix and "T" denotes the transpose. Consider (in the case that $m = n$) also the equations

$$A^kXA = A^k \tag{1^k}$$

$$AX = XA \tag{5}$$

where k is the smallest positive integer such that rank A^k = rank A^{k+1}. Let λ be any nonempty subset of $\{1, 2, 3, 4, 5, 1^k\}$. X is called a λ-inverse of A if X satisfies equation (i) for each $i \in \lambda$. In particular, the $\{1, 2, 3, 4\}$-

* Research of S. K. Jain supported by OUR Grant No. 551.
† Department of Mathematics, Ohio University, Athens, Ohio, 45701.
‡ Departments of Mathematics and Computer Science, Wright State University, Dayton, Ohio 45435.
§ Department of Mathematics and Computer Science, Central State University, Wilberforce, Ohio 45384.

inverse of A is the unique Moore–Penrose generalized inverse and is denoted as A^+. The $\{1, 2, 3, 4\}$-inverse which satisfies (5) must be square. A $\{2, 5, 1^k\}$-inverse is called Drazin pseudoinverse. Drazin pseudoinverse of a matrix A is unique and is denoted as A^D. A $\{1, 2, 5\}$-inverse of A (whenever it exists) is called group inverse of A and is denoted as $A^{\#}$. Clearly, if $A^{\#}$ exists then $A^{\#} = A^D$. The characterizations of all nonnegative matrices whose λ-inverse is nonnegative for any subset λ of $\{1, 2, 3, 4\}$ such that $1 \in \lambda$ are given in [3], [4] and [9].

In this paper, our aim is to characterize all nonnegative matrices A whose $\{1, 2, 3, 4, 5\}$-inverse exists and is nonnegative. This is equivalent to the characterization of all nonnegative matrices A whose Moore–Penrose generalized inverse A^+ is nonnegative and is equal to some polynomial $p(A)$ in A with scalar coefficients, see [1, p. 164, Theorem 3 and p. 173, Corollary 2]. Matrices having a nonnegative generalized inverse which is equal to some polynomial in A are of importance in numerical analysis. A nonnegative matrix A may have a nonnegative generalized inverse which is not expressible as a polynomial in A. For example, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then $A^+ = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$ and $AA^+ \neq A^+A$, showing that A^+ cannot be a polynomial in A. Further, if $p(\lambda)$ is a polynomial with scalar coefficients then the matrix equations $X^+ = p(X)$ may not possess any nonnegative nontrivial solution X such that $X^+ \geqslant 0$ (Example 3).

Theorem 2 of this paper characterizes all nonnegative matrices A whose Moore–Penrose generalized inverse is a polynomial in A and is followed by numerical examples which illustrate the characterization obtained in the theorem. This theorem generalizes the known results for nonnegative matrices A whose A^+ is A [2] or some power of A [7]. The generalization of Berman's theorem [2] is obtained in [7] by first obtaining nonnegative mth roots of nonnegative idempotent symmetric matrices which is also of independent interest. However, it does not appear possible to invoke either technique given in [2] or root extraction technique obtained in [7] to study the case when A^+ is an arbitrary polynomial.

To study our present question we first obtain nonnegative solutions each of rank r of simultaneous matrix equations

$$XY = \left[\begin{array}{ccc|c} x_1y_1^T & & 0 & \\ & \ddots & & 0 \\ 0 & & x_ry_r^T & \\ \hline & 0 & & 0 \end{array}\right], \quad XY = \left[\begin{array}{ccc|c} u_1v_1^T & & 0 & \\ & \ddots & & 0 \\ 0 & & u_rv_r^T & \\ \hline & 0 & & 0 \end{array}\right]$$

where x_i, y_i, u_i, and v_i, $1 \leqslant i \leqslant r$ are positive vectors of the same order (Theorem 1). The proof of Theorem 1, among other lemmas, depend on the following two lemmas proved in [7].

LEMMA A [7, Lemma 2] *Let $A, C_1, \ldots, C_n$ be nonnegative matrices such that $AC_i = 0$ $(C_iA = 0)$, $i = 1, \ldots, n$ and $XA + \Sigma_{i=1}^n C_iY_i > 0$ $(AX + \Sigma_{i=1}^n Y_iC_i > 0)$ for some nonnegative matrices X, Y_i, $1 \leqslant i \leqslant n$. Then $A = 0$ or all C_i's are zero.*

LEMMA B [7, Lemma 3] *Let A, B, C, and D be nonnegative matrices of orders $m \times n$, $n \times m$, $n \times m$, and $m \times n$ respectively such that $AC = 0 = DB$ and each entry on the diagonal of $BA + CD$ is nonzero. Then the j-th column of A is zero if and only if the j-th row of B is zero.*

If in addition, $AB = 0$, then $A = 0 = B$.

If a matrix A is a direct sum of matrices A_i, then A_i shall be called summands of A. The diagonal of any matrix shall mean the main diagonal and a vector shall mean a column vector. A matrix $A = (a_{ij})$ will be called 0-symmetric if it satisfies the following: $a_{ij} = 0$ if and only if $a_{ji} = 0$. Clearly, every positive matrix and every symmetric matrix is 0-symmetric. We shall denote the set of all permutations on $\{1, 2, \ldots, n\}$ by S_n. For any matrix A, $|A|$ denotes the determinant of A. For all other terminology and notations the reader is referred to [1].

2. PRELIMINARY RESULTS

LEMMA 1 *A nonnegative solution of simultaneous matrix equations*

$$XY = \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_r \end{bmatrix}, \qquad YX = \begin{bmatrix} Y_1 & & 0 \\ & \ddots & \\ 0 & & Y_r \end{bmatrix}$$

where X_i, Y_i are positive square matrices of the same orders, is of the form

$$X = (A_{ij}), \quad Y = (B_{jk}), \quad 1 \leqslant i, j, k \leqslant r$$

where the matrix blocks A_{ij} and B_{jk} have the following properties:

i) *A_{ii}, B_{ii} are square matrices of the same order as that of X_i.*

ii) *$A_{i\sigma(i)} \neq 0$, $B_{\sigma(i)i} \neq 0$, $A_{ik} = 0 = B_{ki}\ \forall\, k \neq \sigma(i)$, $1 \leqslant i \leqslant r$, for some $\sigma \in S_r$.*

iii) *$A_{i\sigma(i)}B_{\sigma(i)i} = X_i$, $B_{i\sigma-1(i)}A_{\sigma-1(i)i} = Y_i$.*

Proof By partitioning the solutions, X, Y of the above system of equations into matrix blocks appropriately we can assume that $X = (A_{ij})$, $Y = (B_{jk})$

where A_{ii}, B_{ii} are square matrices of the same orders as that of X_i. We now proceed to establish (ii) and (iii). Clearly, by hypothesis

$$X_j = \sum_{k=1}^{r} A_{jk}B_{kj} \qquad \forall\, 1 \leqslant j \leqslant r \tag{6}$$

$$B_{ij}A_{jk} = 0 \qquad \forall\; i \neq k. \tag{7}$$

From Eqs. (6) and (7) and Lemma A, we get

$$X_j = A_{jl}B_{lj} \quad \text{for some } l,\ 1 \leqslant l \leqslant r$$

and

$$A_{jk} = 0 = B_{kj} \qquad \forall\, k \neq l.$$

Thus there is one and only one nonzero block in each row of X and in each column of Y. Since

$$YX = \begin{bmatrix} Y_1 & & 0 \\ & \ddots & \\ 0 & & Y_r \end{bmatrix}$$

there is one and only one nonzero block in each column of X and in each row of Y. This gives a permutation σ in S_r such that $A_{j\sigma(j)} \neq 0$, $B_{\sigma(j)j} \neq 0$ and $A_{jk} = 0 = B_{kj}$ for all $k \neq \sigma(j)$. Clearly, $A_{i\sigma(i)}B_{\sigma(i)i} = X_i$, $B_{i\sigma-1(i)}A_{\sigma-1(i)i} = Y_i$, as desired.

LEMMA 2 *Let*

$$X = \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \qquad Y = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

be nonnegative matrices (not necessarily square) partitioned into blocks of appropriate orders such that

$$XY = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \qquad YX = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where C, D are square matrices of the same orders having no zero entry on the diagonals. Then $F = G = L = M = 0$, $EK = C$, $KE = D$, $HN = 0 = NH$.

Proof Clearly, we have

$$EK+FM = C, \quad ME = KF = MF = 0 \tag{8}$$

$$KE+LG = D, \quad GK = EL = GL = 0. \tag{9}$$

Then Lemma B yields $M = 0 = F$ and $L = 0 = G$. But then we also get $EK = C$, $KE = D$, $NH = 0 = HN$, completing the proof.

Remark 1 As a consequence of the above lemmas we can rederive Theorem 2 in [7]. So, let A^m be a symmetric idempotent matrix. Then by Flor [6] there exists a permutation matrix P such that

$$(PAP^T)(PA^{m-1}P^T) = PA^mP^T = \left[\begin{array}{ccc|c} x_1x_1^T & & 0 & \\ & \ddots & & 0 \\ 0 & & x_rx_r^T & \\ \hline & 0 & & 0 \end{array}\right] = (PA^{m-1}P^T)(PAP^T).$$

Lemma 2 then yields

$$PAP^T = \begin{bmatrix} E & 0 \\ 0 & H \end{bmatrix} \quad \text{where} \quad E^m = \begin{bmatrix} x_1x_1^T & & 0 \\ & \ddots & \\ 0 & & x_rx_r^T \end{bmatrix}, \qquad H^m = 0.$$

Then by the above Lemma 1 and Lemma 6 in [7] we get the desired result.

Remark 2 Indeed if a polynomial $p(A) = \alpha_1 A + \ldots + \alpha_m A^m$, where α_i are scalars, is an idempotent symmetric matrix one can also obtain somewhat similar characterization of A.

THEOREM 1 *Let X, Y be nonnegative matrices (not necessarily square) each of rank r such that*

$$XY = \left[\begin{array}{ccc|c} x_1y_1^T & & 0 & \\ & \ddots & & 0 \\ 0 & & x_ry_r^T & \\ \hline & 0 & & 0 \end{array}\right]$$

and

$$YX = \left[\begin{array}{ccc|c} u_1v_1^T & & 0 & \\ & \ddots & & 0 \\ 0 & & u_rv_r^T & \\ \hline & 0 & & 0 \end{array}\right]$$

where x_i, y_i, u_i, and v_i, $1 \leqslant i \leqslant r$ are positive vectors of the same order (x_i and x_j, $i \neq j$, are not necessarily of the same order). Then

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

such that the following are true:

i) *$A = (A_{ij})$, $B = (B_{kl})$ where the blocks A_{ii} and B_{ii} are square matrices of the same order as that of $x_iy_i^T$, and all A_{ij}, B_{kl} are zero except when $j = \sigma(i)$, $l = \sigma^{-1}(k)$ for some $\sigma \in S_r$, $1 \leqslant i, j, k, l \leqslant r$.*

ii) $A_{i\sigma(i)} = \alpha_i k_i^{-1} x_i v_{\sigma(i)}^T$, *and* $B_{\sigma(i)i} = \beta_i u_{\sigma(i)} y_i^T$, *where* $\alpha_i \beta_i = 1$, $k_i = |v_{\sigma(i)}^T u_{\sigma(i)}| = |y_i^T x_i|$.

iii) *There exists a permutation matrix P such that* PXP^T *is a direct sum of matrices of types (not necessarily all)*

a) $A_{jj} = \alpha_j k_j^{-1} x_j v_j^T$, *where* $\sigma(j) = j$

b)
$$\begin{bmatrix} 0 & C_{12} & 0 & \dots & 0 \\ 0 & 0 & C_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{d-1d} \\ C_{d1} & 0 & 0 & \dots & 0 \end{bmatrix}$$

where

$$C_{ii+1} = A_{j_i j_{i+1}} = \alpha_{j_i} k_{j_i}^{-1} x_{j_i} v_{j_{i+1}}^T \quad \text{if } i < d,$$
$$C_{d1} = A_{j_d j_1} = \alpha_{j_d} k_{j_d}^{-1} x_{j_d} v_{j_1}^T$$

and d is the length of a cycle $(j_1 \dots j_d)$ *occurring in the disjoint decomposition of* σ.

c) *A zero matrix.*

and PYP^T *is a direct sum of matrices of types (not necessarily all)*

a′) $B_{jj} = \beta_j u_j y_j^T$, *where* $\sigma(j) = j$

b′)
$$\begin{bmatrix} 0 & 0 & \dots & 0 & D_{1d} \\ D_{21} & 0 & \dots & 0 & 0 \\ 0 & D_{32} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & D_{dd-1} & 0 \end{bmatrix}$$

where

$$D_{i+1i} = B_{j_{i+1}j_i} = \beta_{j_i} u_{j_{i+1}} y_{j_i}^T \quad \text{if } i < d$$
$$D_{1d} = B_{j_1 j_d} = \beta_{j_d} u_{j_1} y_{j_d}^T$$

and d is the length of a cycle $(j_1 \dots j_d)$ *occurring in the disjoint decomposition of* σ.

(c′) *a zero matrix.*

Proof Set $C = \begin{bmatrix} x_1 y_1 & & 0 \\ & \ddots & \\ 0 & & x_r y_r^T \end{bmatrix}$ and $D = \begin{bmatrix} u_1 v_1^T & & 0 \\ & \ddots & \\ 0 & & u_r v_r^T \end{bmatrix}$.

Then as in Lemma 2, $X = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix}$, $Y = \begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix}$ where $EK = C$, $KE = D$, $HN = 0 = NH$. Since rank $X = r =$ rank $C \leqslant$ rank $E \leqslant$ rank X, $H = 0$. Similarly, $N = 0$. Also by Lemma 1, $E = (A_{ij})$, $K = (B_{jk})$ where the matrix blocks satisfy (i). Since E (or K) is of rank r and there is one and only one nonzero block in each row and column of E (or K), each $A_{i\sigma(i)}$ (or $B_{\sigma(i)i}$) is of rank 1, $1 \leq i \leq r$. But then $A_{i\sigma(i)} = a_i b_i^T$, and $B_{\sigma(i)i} = c_i d_i^T$ where a_i, b_i, c_i, and d_i are nonzero vectors. Since $A_{i\sigma(i)}B_{\sigma(i)i}$ and $B_{\sigma(i)i}A_{i\sigma(i)}$ are positive matrices we can indeed choose a_i, b_i, c_i, and d_i as positive vectors. In order to prove (ii) we note that

$$A_{i\sigma(i)}B_{\sigma(i)i} = x_i y_i^T \tag{10}$$

and

$$B_{i\sigma^{-1}(i)}A_{\sigma^{-1}(i)i} = u_i v_i^T. \tag{11}$$

Therefore from (10) $a_i(b_i^T c_i)d_i^T = x_i y_i^T$ whence $a_i = \mu_i\, |b_i^T c_i|^{-1} x_i$ and $d_i = \mu_i^{-1} y_i$ where μ_i is an arbitrary positive number. Similarly from (11) we get $c_{\sigma^{-1}(i)} = \lambda_i u_i$, and $b_{\sigma^{-1}(i)} = \lambda_i^{-1}\, |d_{\sigma^{-1}(i)}^T a_{\sigma^{-1}(i)}|^{-1} v_i$ where λ_i is an arbitrary positive number. Thus

$$A_{i\sigma(i)} = \mu_i(\lambda_{\sigma(i)}|b_i^T c_i|\, |d_i^T a_i|)^{-1} x_i v_{\sigma(i)}^T \tag{12}$$

$$B_{\sigma(i)i} = \lambda_{\sigma(i)}\mu_i^{-1} u_{\sigma(i)} y_i^T. \tag{13}$$

If we put $k_i = |b_i^T c_i|\, |d_i^T a_i|$, then the relations (10)–(13) give

$$k_i = |v_{\sigma(i)}^T u_{\sigma(i)}| = |y_i^T x_i|.$$

Also by setting $\alpha_i = \mu_i \lambda_{\sigma(i)}^{-1}$ and $\beta_i = \lambda_{\sigma(i)}\mu_i^{-1}$, we obtain (ii).

To prove (iii) we recall that any permutation $\sigma \in S_n$ can be expressed as a unique product of disjoint cycles, i.e. $\sigma = (i_1 \ldots i_{d_1})\,(j_i \ldots j_{d_2}) \ldots$, where $\sigma(i_1) = i_2, \ldots, \sigma(i_{d_1}) = i_1$ and similarly for j's etc. It is clear that corresponding to each cycle of length d, there is a $d \times d$ minor of X with $A_{i\sigma(i)}$, $A_{\sigma(i)\sigma^2(i)}, \ldots, A_{\sigma^{d-1}(i)i}$ as its nonzero entries (X being regarded as $(r+1) \times (r+1)$ matrix whose entries are blocks). By interchanging rows and columns of X suitably we can bring the rows and columns of each such minor adjacent to each other. Also this interchange of rows and columns transforms X into PXP^T for some permutation matrix P. This proves that PXP^T is a direct sum of types (a), (b) and (c) stated in the theorem. More specifically, type (a) shall correspond to cycles of length 1 and type (b) shall correspond to cycles of length $d > 1$. Similarly, we can prove that PYP^T is a direct sum of types (a′), (b′) and (c′).

3. MAIN THEOREM

THEOREM 2 *Let A, A^+ be nonnegative matrices such that $A^+ = p(A)$ where $p(A) = \alpha_1 A^{m_1} + \ldots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i \geq 0$. Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following three types (not necessarily all)*:

I) $\beta x x^T$, *where* $\beta > 0$, $\Sigma_{m_i} \alpha_i \beta^{m_i+1} = 1$, *and* x *is a positive unit vector*

II)

$$\begin{bmatrix} 0 & \beta_{12}x_1x_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2x_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_{d-1\,d}x_{d-1}x_d^T \\ \beta_{d1}x_dx_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where x_i *are positive unit vectors,* x_i *and* x_j, $i \neq j$, *are not necessarily of the same order, and* $\beta_{12}, \beta_{23}, \ldots, \beta_{d1}$ *are arbitrary positive numbers with* $d \neq 1$ *such that their product* $\beta_{12}\beta_{23}\ldots\beta_{d1}$ *is a common root of the following system of at most d equations in t*

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1 \tag{14}$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0 \qquad k \in \{1, \ldots, d-1\} \tag{15}$$

where the summation in each of the above equations runs over all those m_i *for which* $d \mid (m_i+1-k)$, $k = 0, 1, \ldots, d-1$, *with the convention that if there is no* m_i *for which* $d \mid (m_i+1-k)$, $k \in \{1, \ldots, d-1\}$ *then the corresponding equation is absent. (Naturally, the possible values of d are divisors of* m_i+1. *Among these divisors we shall discard those divisors d for which the above system of equations has no common positive solution.)*

III) *A zero matrix.*

In particular, if all $\alpha_i > 0$ then β in type (I) and the product $\beta_{12}\beta_{23}\ldots\beta_{d1}$ in type (II) are unique. Further, in this case the positive integer d, i.e. the rank of a matrix of type (II), must divide each m_i+1.

Conversely, if A is a nonnegative matrix and P is a permutation matrix such that PAP^T is a direct sum of matrices of the following three types (not necessarily all).

I′) $\beta x x^T$, $\beta > 0$, x is some positive unit vector.

II′)

$$\begin{bmatrix} 0 & \beta_{12}x_1x_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2x_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_{d-1\,d}x_{d-1}x_d^T \\ \beta_{d1}x_dx_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $\beta_{ij} > 0$, x_i are positive unit vectors.

III′) *A zero matrix.*

Then $A^+ \geqslant 0$ and is equal to some polynomial in A with scalar coefficients.

Proof Let A and A^+ be nonnegative matrices such that $A^+ = p(A)$ where $p(A) = \alpha_1 A^{m_1} + \ldots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i > 0$. Then $AA^+ = A^+A$.

Also since AA^+ is a symmetric idempotent matrix, by Flor [6], there exists a permutation matrix Q such that

$$(QAQ^T)(QA^+Q^T)=(QAA^+Q^T)=\left[\begin{array}{ccc|c} x_1x_1 & & 0 & \\ & \cdot & & \\ & & \cdot & 0 \\ & & \cdot & \\ 0 & & x_rx_r^T & \\ \hline & 0 & & 0 \end{array}\right]=(QA^+Q^T)(QAQ^T)$$

for some integer r, where x_i are positive unit vectors. We may note that rank (QAQ^T) = rank (QA^+Q^T) = rank $(QAA^+Q^T) = r$. We now set $X = QAQ^T$ and $Y = QA^+Q^T$ and invoke Theorem 1 to obtain a permutation σ in S_r and a permutation matrix P such that PAP^T is a direct sum of matrices of the following three types (not necessarily all)

a) $\beta_j x_j x_j^T$ where $\beta_j > 0$, $j = \sigma(j)$, x_j is a positive vector

b)
$$\begin{bmatrix} 0 & C_{12} & 0 & \dots & 0 \\ 0 & 0 & C_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{d-1d} \\ C_{d1} & 0 & 0 & \dots & 0 \end{bmatrix}$$

where

$$C_{ii+1} = \beta_{j_i} x_{j_i} x_{j_{i+1}}^T, \qquad i < d,\ \beta_{j_i} > 0$$
$$C_{d1} = \beta_{j_d} x_{j_d} x_{j_1}^T, \qquad \beta_{j_d} > 0$$

d is the length of a cycle $(j_1 \dots j_d)$ occurring in the disjoint decomposition of σ, and x_i's are positive unit vectors.

c) A zero matrix.

Also $AA^+A = A$ implies

$$\alpha_1 A^{m_1+2} + \dots + \alpha_k A^{m_k+2} = A. \tag{16}$$

Clearly, all summands S of PAP^T must satisfy equation (16), i.e.

$$\alpha_1 S^{m_1+2} + \dots + \alpha_k S^{m_k+2} = S. \tag{17}$$

Let $S = \beta xx^T$ be a summand of type (a) for some positive number β and unit positive vector x. Since xx^T is an idempotent matrix Eq. (17) implies

$$\alpha_1 \beta^{m_1+1} + \dots + \alpha_k \beta^{m_k+1} = 1. \tag{18}$$

Next let

$$S = \begin{bmatrix} 0 & \beta_{12}x_1x_2^T & 0 & \dots & 0 \\ 0 & 0 & \beta_{23}x_2x_3^T & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \beta_{d-1d}x_{d-1}x_d^T \\ \beta_{d1}x_dx_1^T & 0 & 0 & \dots & 0 \end{bmatrix}$$

be a summand of type (b) for some positive numbers β_{ij} and unit positive vectors x_i. Then Eq. (17) implies

$$\alpha_1 S^{m_1+1+d} + \ldots + \alpha_k S^{m_k+1+d} = S^d. \tag{19}$$

Clearly, for all $1 \leqslant k \leqslant d-1$ we have

$$S^k = \begin{bmatrix} 0 & \ldots & 0 & w_{1k+1}x_1x_{k+1}^T & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & w_{2k+2}x_2x_{k+2}^T & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & w_{d-kd}x_{d-k}x_d^T \\ w_{d-k+1\,1}x_{d-k+1}x_1^T & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & w_{dk}x_dx_k^T & 0 & 0 & \ldots & 0 \end{bmatrix}$$

where $w_{ij} = \beta_{ii+1} \ldots \beta_{j-1j}$ for $i < j$, $w_{ij} = \beta_{ii+1} \ldots \beta_{d-1d}\beta_{d1} \ldots \beta_{j-1j}$ for $i > j$, $i \neq d$, and $w_{dj} = \beta_{d1} \ldots \beta_{j-1j}$, also

$$S^d = (\beta_{12}\beta_{23} \ldots \beta_{d1}) \begin{bmatrix} x_1x_1^T & & 0 \\ & \ddots & \\ 0 & & x_dx_d^T \end{bmatrix}.$$

Hence $S^q = (\beta_{12} \ldots \beta_{d1})^p S^k$, where $q = pd+k$, $1 \leqslant k \leqslant d$. Now by Eq. (19) we have

$$\alpha_1 S^{m_1+1} + \ldots + a_k S^{m_k+1} = \begin{bmatrix} x_1x_1^T & & 0 \\ & \ddots & \\ 0 & & x_dx_d^T \end{bmatrix}.$$

By comparing the entries in the first row of above matrix equation we get

$$\sum_{d|(m_i+1)} \alpha_i(\beta_{12} \ldots \beta_{d1})^{(m_i+1)/d} = 1 \tag{20}$$

and

$$\sum_{d|(m_i+1-k)} \alpha_i(\beta_{12} \ldots \beta_{d1})^{(m_i+1-k)/d} = 0, \qquad i \leqslant k \leqslant d-1 \tag{21}$$

with the convention that if there is no m_i such that $d \mid (m_i+1-k)$ for some $k \in \{1, 2, \ldots, d-1\}$ then the corresponding equation is absent.

In particular, let us assume all $\alpha_i > 0$. Then by Descarte's rule of signs in the theory of algebraic equations and intermediate value theorem in analysis β is the only positive root of the equation

$$\alpha_1 t^{m_1+1} + \ldots + \alpha_k t^{m_k+1} = 1.$$

Therefore, β in type (I) is unique. Similarly, the product $\beta_{12}\beta_{23} \ldots \beta_{d1}$ in type (II) is unique. Let d be the rank of a matrix in type (II). Then d must divide each (m_i+1). Otherwise, let d does not divide some m_i+1. Then there exists an integer $k \in \{1, 2, \ldots, d-1\}$ such that $d \mid (m_i+1-k)$. Hence by Eq. (21)

$$\sum_{\substack{d \neq 1 \\ d \mid m_i+1-k}} \alpha_i(\beta_{12} \ldots \beta_{d1})^{((m_i+1-k)/d)} = 0$$

which is impossible since $\alpha_i > 0$. Hence d divides each (m_i+1) as desired. This completes the proof.

To prove the converse, we first note that for each of the types (I′), (II′) and (III′) of matrices S, $S^+ \geqslant 0$ and $S^+S = SS^+$. Thus if PAP^T is a direct sum of matrices of types (I′), (II′) and (III′), then $(PAP^T)^+ \geqslant 0$ and $(PAP^T)(PAP^T)^+ = (PAP^T)^+(PAP^T)$, that is, $A^+ \geqslant 0$ and $AA^+ = A^+A$. This implies $A^+ = A^D$, a polynomial in A, completing the proof.

Remark 3 Let A be nonnegative matrix and $p(A) = \alpha_1 A^{m_1} + \ldots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i \geqslant 0$ such that $p(A) \geqslant 0$, $Ap(A)$ is 0-symmetric, $Ap(A)A = A$, and rank A = rank $p(A)$. Then similar arguments as in Theorem 2 yield that there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following three types (not necessarily all)

i) βxy^T, where x and y are positive vectors with $y^Tx = 1$, and β is some positive number satisfying $\Sigma_{m_i} \alpha_i \beta^{m_i+1} = 1$

ii)
$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \ldots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \beta_{d-1d}x_{d-1}y^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \ldots & 0 \end{bmatrix}$$

where x_i and y_i are positive vectors of the same order with $y^Tx_i = 1$, x_i and x_j, $i \neq j$, are not necessarily of the same order, and other restrictions on the choice of d and β's remain the same as in Theorem 2.

iii) A zero matrix.

4. NUMERICAL EXAMPLES

Now we proceed to give numerical examples to illustrate our Theorem 2.

Example 1 Let A be a nonnegative square matrix such that

$$A^+ = -2A^2+2A^4-5A^7+6A^{11}.$$

By Theorem 2, there exists a permutation matrix P such that PAP^T is a direct sum of matrices of types (I), (II) or (III). We first determine type (I). This type of matrices are of the form βxx^T where $-2\beta^3+2\beta^5-5\beta^8+6\beta^{12} = 1$, i.e. β is a root of $f(t) = 6t^{12}-5t^8+2t^5-2t^3-1$. Now

$$f(t) = (t-1)g(t),$$

where $g(t) = 5t^8(t^3+t^2+t+1)+2t^3(t+1)+(t^{11}+t^{10}+\ldots+1)$. Clearly, $g(t)$ has no positive solution. Hence $\beta = 1$ is the only positive solution of $f(t)$, and thus xx^T is the only possible form of matrices in type (I).

Next we determine the matrices in type (II). Recall that the type (II) contains matrices of rank d ($d \neq 1$) and the possible values of d are divisors of (m_i+1). Here, $m_1 = 2$, $m_2 = 4$, $m_3 = 7$, $m_4 = 11$. So possible values of d ($d \neq 1$) are 2, 3, 4, 5, 6, 8, and 12. Among these values of d we discard those values for which the system of Eqs. (14) and (15), that is,

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1 \tag{14}$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0, \qquad k \in \{1, \ldots, d-1\} \tag{15}$$

has no common solution. (Here $\alpha_1 = -2$, $\alpha_2 = 2$, $\alpha_3 = -5$, $\alpha_4 = 6$). We show below that d cannot be equal to 3, 4, 5, 6, 8, 12. If $d = 3$, then the above system of equations becomes

$$\begin{aligned} -2t+6t^4 &= 1 \\ 2t-5t^2 &= 0 \end{aligned}$$

which clearly do not have a common positive solution, showing $d = 3$ is not possible.

If $d = 4$, then the above system of equations becomes

$$\begin{aligned} -5t^2+6t^3 &= 1 \\ 2t &= 0 \\ -2t^0 &= 0. \end{aligned}$$

Again there is no common solution to the above system, proving that d cannot be equal to 4. We can dispose of other values of d similarly. Hence the only possible value of d is 2. In this case, the system of equations becomes

$$\begin{aligned} -5t^4+6t^6 &= 1 \\ -2t+2t^2 &= 0. \end{aligned}$$

Clearly, $t = 1$ is the only common positive solution. Thus the matrices of type (II) will be of the form

$$\begin{pmatrix} 0 & ax_1x_2^T \\ bx_2x_1^T & 0 \end{pmatrix},$$

where $a > 0$, $b > 0$ and $ab = 1$, and x_1, x_2 are positive unit vectors. Therefore, PAP^T is a direct sum of matrices of the form (not necessarily all):

i) xx^T, where x is a positive unit vector

ii) $$\begin{pmatrix} 0 & ax_1x_2^T \\ bx_2x_1^T & 0 \end{pmatrix},$$

$a > 0$, $b > 0$, $ab = 1$, and x_1, x_2 are positive unit vectors.

iii) A zero matrix.

We may remark that in this case $A^+ = A$, i.e. the given polynomial $6A^{11}-5A^7+2A^4-2A^2$ reduces to A.

Example 2 Let A be a nonnegative matrix with a nonnegative generalized inverse

$$A^+ = 2A^{15} - (511/16)A^7 - 8A^6 + 32A^2.$$

The matrices of type (I) are of the form βxx^T, where β is a positive root of $h(t) = 2t^{16} - (511/16)t^8 - 8t^7 + 32t^3 - 1$. Now by the Descarte's rule of signs $h(t)$ can have at most three positive roots. But by intermediate value theorem $h(t)$ has at least three (hence exactly three) positive roots in the intervals (0.31, 0.32), (0.95, 0.96) and (1.41, 1.42).

Next to determine matrices of type (II) we proceed as in example 1 and discard all the divisors d of $(m_i + 1)$ except when $d = 2, 4$. For $d = 2$, the system of Eqs. (14) and (15) becomes

$$2t^8 - (511/16)t^4 = 1$$
$$-8t^3 + 32t = 0.$$

The only positive common solution of above equations is $t = 2$. For $d = 4$, the system of Eqs. (14) and (15) becomes

$$2t^4 - (511/16)t^2 = 1$$
$$-8t + 32 = 0.$$

The only positive common solution of above equations is $t = 4$. Thus PAP^T is a direct sum of matrices of the form (not necessarily all):

i) βxx^T, where β is a positive root of $h(t)$ and x is a positive unit vector

ii) $$\begin{pmatrix} 0 & \alpha x_1 x_2^T \\ \beta x_2 x_1^T & 0 \end{pmatrix},$$

$\alpha > 0$, $\beta > 0$, $\alpha\beta = 2$, and x_1, x_2 are positive unit vectors.

iii) $$\begin{bmatrix} 0 & \alpha y_1 y_2^T & 0 & 0 \\ 0 & 0 & \beta y_2 y_3^T & 0 \\ 0 & 0 & 0 & \gamma y_3 y_4 \\ \delta y_4 y_1^T & 0 & 0 & 0 \end{bmatrix},$$

$\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $\alpha\beta\gamma\delta = 4$, and y_1, y_2, y_3, y_4 are positive unit vectors.

iv) A zero matrix.

In the next example we show that if $p(\lambda)$ is a polynomial with scalar coefficients then the matrix equation $X^+ = p(X)$ may not possess any nonnegative nontrivial solution X such that $X^+ \geqslant 0$.

Example 3 Consider

$$p(\lambda) = -2\lambda^{15} - 32\lambda^7 - 8\lambda^6 + \lambda^2.$$

In case $X^+ = p(X)$ has a solution containing a summand of type (I), then

$$f(t) = -2t^{16} - 32t^8 - 8t^7 + t^3 - 1$$

must have a positive root which is not true. To look for solutions of $X^+ = p(X)$ in matrices of type (II), we proceed as in example 1 and 2. It can be verified that no divisor d of (m_i+1) is acceptable. Thus the only possible solution is the trivial solution $X = 0$.

We close this example by a remark that if all the coefficients of $p(\lambda)$ are positive then $X^+ = p(X)$ always possesses a nontrivial nonnegative solution.

Remarks 1) Emilie Haynsworth and J. R. Wall have in their paper "Group inverses of certain nonnegative matrices", to appear in *Linear Algebra and Applications*, among others, characterized nonnegative matrices A with $A^{\#} = A^k$, k is some positive integer.

2) In our other paper "Decomposition of nonnegative group-monotone matrices", submitted for publication, we have obtained a decomposition of nonnegative matrices having nonnegative group inverses. This decomposition characterizes all nonnegative matrices with nonnegative group inverses and provides a new approach to the solutions of problems relating to such matrices.

References

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, New York (1974).

[2] A. Berman, Nonnegative matrices which are equal to their generalized inverse, *Linear Algebra and Appl.* **9** (1974), 261–265.

[3] A. Berman and R. J. Plemmons, Monotonicity and the generalized inverse, *SIAM J. Appl. Math* **22** (1972), 155–161.

[4] A. Berman and R. J. Plemmons, Inverses of nonnegative matrices, *Linear and Multilinear Algebra* **2** (1974), 161–172.

[5] R. E. Cline, Inverses of rank invariant powers of a matrix, *SIAM J. Numer. Anal.* **5** (1968), 182–197.

[6] P. Flor, On groups of nonnegative matrics, *Compositio Math.* **21** (1969), 376–382.

[7] S. K. Jain, V. K. Goel and Edward K. Kwak, Nonnegative m-th roots of nonnegative 0-symmetric idempotent matrices, *Linear Algebra and Applications*, to appear.

[8] R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.* **51** (1955), 406–413.

[9] R. J. Plemmons and R. E. Cline, The generalized inverse of a nonnegative Matrix, *Proc. Amer. math. Soc.* **31** (1972), 46–50.