Nonnegative mth Roots of Nonnegative 0-Symmetric Idempotent Matrices

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ABSTRACT

Nonnegative mth roots of nonnegative 0-symmetric idempotent matrices have been characterized. As an application, a characterization of nonnegative matrices A whose Moore-Penrose generalized inverse $A^{\dagger}$ is some power of A is obtained, thus yielding some well-known theorems.

1. INTRODUCTION

Let A be an $m\times n$ real matrix. Consider the Penrose [8] equations

$$AXA=A, \tag{1}$$

$$XAX=X, \tag{2}$$

$$(AX)^T=AX, \tag{3}$$

$$(XA)^T=XA, \tag{4}$$

where X is an $n\times m$ real matrix and T denotes the transpose. Consider also the equations

$$A^kXA=A^k, \tag{1^k}$$

$$AX=XA, \tag{5}$$

where k is some positive integer.

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For a rectangular matrix A and for a nonempty subset λ of $\{1,2,3,4\}$, X is called a λ-inverse of A if X satisfies Eq. (i) for each $i \in \lambda$. In particular, the $\{1,2,3,4\}$-inverse of A is the unique Moore-Penrose generalized inverse. The unique solution X of (2), (1^k), and (5) is a square matrix called the Drazin inverse of A, where k is the smallest positive integer such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$.

A matrix $A=(a_{ij})$ is called 0-symmetric if $a_{ij}=0$ implies $a_{ji}=0$. Thus every symmetric matrix and every positive matrix is 0-symmetric. If a matrix A is a direct sum of matrices A_i, then A_i will be called summands of a.

The problem of finding the mth roots of any matrix A is an important classical problem (see Gantmacher [4], Chapter 8). In this paper our aim is to study the nonnegative mth roots of nonnegative 0-symmetric idempotent matrices. Theorem 1 of this paper reduces the study of the nonnegative mth roots of any nonnegative 0-symmetric idempotent matrix to the nonnegative kth roots of matrices of the form xy^T (x, y positive vectors with $y^Tx=1$), and to the nonnegative solution of a system of simultaneous equations of the type $X_1X_2\dots X_d = x_1 y_1^T, \dots, X_dX_1\dots X_{d-1} = x_d y_d^T$ (x_i, y_i positive vectors with $y_i^T x_i = 1$). Clearly, xy^T is the only nonnegative kth root of rank 1 of the positive idempotent matrix xy^T. However, the nonnegative kth roots of ranks greater than 1 are not considered, and it remains open to determine such roots. In Sec. 4, we use the reduction obtained in Theorem 1 to characterize the nonnegative matrices A such that A^k is 0-symmetric and $A^{k+1}=A$ for some positive integer k. This, in particular, determines all nonnegative matrices A whose generalized inverse $A^\dagger$ is some power of A. This result generalizes the recent results of Harary and Minc [5] for nonnegative matrices A with $A^{-1}=A$ and that of Berman [1] for nonnegative matrices A with $A^\dagger = A$.

1.1. Notation and Conventions

S_n:	the group of permutations on $\{1,2,\dots,n\}$.
$A^\dagger$:	Moore-Penrose generalized inverse.
A^D:	Drazin inverse.
$A \geqslant 0$:	a matrix with nonnegative entries.
$A > 0$:	a matrix with positive entries.
$\mathcal{Q}$:	a set of nonnegative matrices.
$\sqrt[m]{\mathcal{Q}}$:	$\{X \mid X^m \in \mathcal{Q}\}$.
$+\sqrt[m]{\mathcal{Q}}$:	$\{X \geqslant 0 \mid X^m \in \mathcal{Q}\}$.
$C_{pq}^{(i)}$:	the (p,q)th block of the ith power of a partitioned matrix C.

The diagonal of any square matrix shall mean the main diagonal. By a vector we shall mean a column vector.

2. MAIN RESULTS

THEOREM 1. *Let $\mathfrak{B}$ be the set of all nonnegative 0-symmetric idempotent matrices. Then $A \in +\sqrt[m]{\mathfrak{B}}$ if and only if there exists a permutation matrix P such that PAP^T is a direct sum of square matrices of the following (not necessarily all) three types:*

(I) C_{11}, *where* $C_{11}^m = xy^T$, *for some positive vectors x and y such that* $y^Tx = 1$.

(II)

$$\begin{bmatrix} 0 & C_{12} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & C_{23} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{d-1d} \\ C_{d1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $(C_{12}C_{23}\dots C_{d1})^{m/d} = x_1 y_1^T, \dots, (C_{d1}C_{12}\dots C_{d-1d})^{m/d} = x_d y_d^T$; x_i, y_i *are positive vectors of the same order with* $y_i^T x_i = 1$; x_i *and* x_j, $i \neq j$, *are not necessarily of the same order*; $d|m$, $d \neq 1$; *and the zeros on the diagonal are square matrices of appropriate orders.*

(III)

$$\begin{bmatrix} 0 & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1l} \\ 0 & 0 & C_{23} & \cdot & \cdot & \cdot & C_{2l} \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{l-1l} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $l \leqslant m$, *the* C_{ij}*'s are nonnegative matrices of appropriate orders, and the zeros on the diagonal are square matrices.*

THEOREM 2. *Let $\mathfrak{B}$ be the set of all nonnegative symmetric idempotent matrices. Then $A \in +\sqrt[m]{\mathfrak{B}}$ if and only if there exists a permutation matrix P such that PAP^T is a direct sum of square matrices of the following (not necessarily all) three types:*

(I) C_{11}, *where* $C_{11}^m = xx^T$ *and x is a positive unit vector.*

(II)

$$\begin{bmatrix} 0 & C_{12} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & C_{23} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{d-1d} \\ C_{d1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $(C_{12}C_{23}\dots C_{d1})^{m/d}=x_1x_1^T,\dots,(C_{d1}C_{12}\dots C_{d-1d})^{m/d}=x_dx_d^T$; *the* x_i*'s are positive unit vectors (not necessarily of the same order);* $d|m$, $d\neq 1$; *and the zeros on the diagonal stand for the square matrices of appropriate orders.*

(III)

$$\begin{bmatrix} 0 & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1l} \\ 0 & 0 & C_{23} & \cdot & \cdot & \cdot & C_{2l} \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{l-1l} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $l\leqslant m$, *the* C_{ij}*'s are nonnegative matrices of appropriate orders, and the zeros on the diagonal stand for the square matrices.*

3. PRELIMINARY RESULTS AND PROOFS OF THEOREMS 1 AND 2

In order to prove Theorems 1 and 2 we shall prove a few lemmas. We first recall that if A, B are nonnegative matrices of orders $m\times n$, $n\times k$, respectively, such that $AB=0$, then for any i, $1\leqslant i\leqslant n$, the ith column of A and the ith row of B cannot both be nonzero. We now prove

LEMMA 1. *Let* A, C *be nonnegative (not necessarily square) matrices such that* $AC=0$, *and* $XA+CY>0$ *for some matrices* X *and* Y *(not necessarily nonnegative). Then* $A=0$ *or* $C=0$.

Proof. Assume $A\neq 0$, $C\neq 0$. Then $AC=0$ implies that there exists a zero column of A (hence of XA) and a zero row of C (hence of CY). But then $XA+CY$ cannot be positive, a contradiction. ■

LEMMA 2. *Let $A, C_1,\dots,C_n$ be nonnegative matrices such that $AC_i=0$ $(C_iA=0)$, $i=1,\dots,n$, and $XA+\Sigma_{i=1}^n C_iY_i>0$ $(AX+\Sigma_{i=1}^n Y_iC_i>0)$ for some nonnegative matrices X, Y_i, $1\leqslant i\leqslant n$. Then $A=0$ or all C_i's are zero.*

Proof. Observe $A(\Sigma_{i=1}^n C_i)=0$ and $XA+(\Sigma_{i=1}^n C_i)(\Sigma_{i=1}^n Y_i)>0$, and apply Lemma 1. ∎

LEMMA 3. *Let A, B, C, and D be nonnegative matrices of orders $m\times n$, $n\times m$, $n\times m$, and $m\times n$, respectively, such that $AC=0=DB$ and each entry on the diagonal of $BA+CD$ is nonzero. Then the jth column of A is zero if and only if the jth row of B is zero.*

If in addition, $AB=0$, then $A=0=B$.

Proof. If A, B, C, or D is zero, then the proof is trivial. So assume each of the matrices A, B, C, and D is not zero. Let the jth column of A be zero. Then the jth column of BA is zero. Since the diagonal of $BA+CD$ is nonzero, this implies that the jth column of CD cannot be zero. Hence the jth column of D cannot be zero. But then $DB=0$ implies that the jth row of B is zero. The converse can be proved similarly.

The last statement follows trivially. ∎

LEMMA 4. *Let $\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}$ be a nonnegative matrix such that the diagonal blocks are square matrices and each entry on the diagonal of D is nonzero. Then*

$$+\sqrt[m]{\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}}=\begin{pmatrix} +\sqrt[m]{D} & 0\\ 0 & +\sqrt[m]{0}\end{pmatrix}.$$

Proof. Let

$$C=\begin{pmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\in+\sqrt[m]{\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}}.$$

Then

$$CC^{m-1}=C^{m-1}C=\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}$$

implies

$$C_{11}C_{11}^{(m-1)}+C_{12}C_{21}^{(m-1)}=D, \qquad C_{11}^{(m-1)}C_{12}=C_{21}^{(m-1)}C_{11}=C_{21}^{(m-1)}C_{12}=0,$$

and

$$C_{11}^{(m-1)}C_{11}+C_{12}^{(m-1)}C_{21}=D, \qquad C_{11}C_{12}^{(m-1)}=C_{21}C_{11}^{(m-1)}=C_{21}C_{12}^{(m-1)}=0.$$

Thus, by Lemma 3, $C_{12}=0=C_{21}$. Then $C_{11}^m=D$ and $C_{22}^m=0$. Hence

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \begin{pmatrix} +\sqrt[m]{D} & 0 \\ 0 & +\sqrt[m]{0} \end{pmatrix},$$

completing the proof. ∎

LEMMA 5. *Let*

$$C=\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} \in +\sqrt[m]{\begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix}},$$

where C_{ij} is a nonnegative matrix of order $l_i \times l_j$ and A_i is a positive square matrix of order l_i, $1 \leqslant i \leqslant n$. Then there exists a $\sigma \in S_n$ such that

(a) $C_{j\sigma(j)} \neq 0$, $C_{jk}=0$ $\forall k \neq \sigma(j)$, $j=1,\dots,n$.

(b) $C_{j\sigma(j)}C_{\sigma(j)\sigma^2(j)}\cdots C_{\sigma^{m-1}(j)j}=A_j$. *[Equivalently, if d_j is the smallest positive integer such that $\sigma^{d_j}(j)=j$, then $(C_{j\sigma(j)}\cdots C_{\sigma^{d_j-1}(j)j})^{m/d_j}=A_j$.]*

(c) $\sigma^m=I$, *the identity permutation.*

(d) *There exists a permutation matrix P such that PCP^T is a direct sum of square matrices of the types (I) or (II) described in Theorem 1.*

Proof. Since

$$C^m=\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix},$$

we get

$$C_{ik}^{(m-1)}C_{kj}=0=C_{ik}C_{kj}^{(m-1)} \qquad \text{for all} \quad i\neq j \tag{6}$$

and

$$A_j=C_{j1}C_{1j}^{(m-1)}+\cdots+C_{jl}C_{lj}^{(m-1)}+\cdots+C_{jn}C_{nj}^{(m-1)}. \tag{7}$$

Assume $C_{jl}C_{lj}^{(m-1)}\neq 0$. Then $C_{lj}^{(m-1)}\neq 0$, and thus by (6), (7), and Lemma 2, $C_{jk}=0\ \forall k\neq l$. Note that $C_{jl}\neq 0$ and $A_j=C_{jl}C_{lj}^{(m-1)}$. Hence each row of C has one and only one nonzero block. Since the matrix C^m has no zero column, the same is true for the matrix C. Therefore, there is one and only one nonzero block in each row and in each column of C. This determines a permutation $\sigma\in S_n$ such that

$$C_{j\sigma(j)}\neq 0, \quad C_{jk}=0\ \forall k\neq\sigma(j), \qquad j=1,\dots,n. \tag{8}$$

Then from (7) and (8), $A_j=C_{jp_1}C_{p_1p_2}\cdots C_{p_{m-1}j}$. But then $C_{jp_1}\neq 0$, $C_{p_1p_2}\neq 0,\dots,C_{p_{m-1}j}\neq 0$ imply $p_1=\sigma(j)$, $p_2=\sigma^2(j),\dots,$ $p_{m-1}=\sigma^{m-1}(j)$, and $j=\sigma^m(j)$. Hence $\sigma^m=I$, the identity permutation, proving (b) and (c).

Since any permutation σ can be expressed as a product of disjoint cycles, (d) follows by straightforward computations. ∎

LEMMA 6. *Let* $0\neq C\in+\sqrt[m]{0}$, *where* 0 *is a square matrix of order* n. *Then there exists a permutation matrix* P *such that*

$$PCP^T=\begin{bmatrix} 0 & C_{12} & C_{13} & \cdot & \cdot & \cdot & C_{1l}\\ 0 & 0 & C_{23} & \cdot & \cdot & \cdot & C_{2l}\\ \vdots & \vdots & & & & & \vdots\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{l-1l}\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0\end{bmatrix},$$

where $l\leqslant m$, *the* 0*'s on the diagonal stand for square matrices, and the* C_{ij}*'s are nonnegative matrices of appropriate orders.*

Proof. If $C=0$ then the proof is trivial. So assume $C\neq 0$. Then $m>1$. We shall prove this result by induction on m. So suppose $m=2$. Then $C^2=0$ implies that there exists a $\sigma\in S_n$ and $1\leqslant r\leqslant n$ such that $\sigma(1)$th$,\dots,\sigma(r)$th rows and $\sigma(r+1)$th,...,$\sigma(n)$th columns of C are zero. This gives a permutation matrix P such that PCP^T is of the required form. We now assume that the result is true for $m=k-1$ and prove the result for $m=k$. Since $C^k=0$ we have $(C^{k-1})^2=0$. By induction there exists a permutation matrix P_1 such that

$$P_1C^{k-1}P_1^T=\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}.$$

Without loss of generality, we can assume that each row of the matrix block D is nonzero. Let

$$P_1CP_1^T=\begin{pmatrix}A & E\\ B & F\end{pmatrix}.$$

Then $P_1C^kP_1^T=0$ gives

$$\begin{pmatrix}A & E\\ B & F\end{pmatrix}\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}=0.$$

This implies $AD=0=BD$. But since no row of D is zero, we get $A=0=B$. Thus

$$P_1CP_1^T=\begin{pmatrix}0 & E\\ 0 & F\end{pmatrix}.$$

Then

$$\begin{pmatrix}0 & E\\ 0 & F\end{pmatrix}^{k-1}=\left(P_1CP_1^T\right)^{k-1}=\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix}$$

implies $F^{k-1}=0$. Again by the induction assumption, there exists a permutation matrix P_2 such that

$$P_2FP_2^T=\begin{bmatrix}0 & F_{12} & F_{13} & \cdot & \cdot & \cdot & F_{1q}\\ 0 & 0 & F_{23} & \cdot & \cdot & \cdot & F_{2q}\\ \vdots & \vdots & & & & & \vdots\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & F_{q-1q}\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0\end{bmatrix}, \qquad q\leqslant k-1.$$

Then

$$P=\begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix} P_1$$

is a desired permutation matrix. ■

Proof of Theorem 1.

"Only if" part.

Let $A \in \sqrt[m]{\mathcal{B}}$. Then there exists a matrix $B \in \mathcal{B}$ such that $A^m = B$. Since B is a 0-symmetric idempotent matrix, there exists, by Flor [3], a permutation matrix P such that

$$PBP^T = \left[\begin{array}{ccc|c} A_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & A_s & \\ \hline & 0 & & 0 \end{array}\right],$$

where $A_i = x_i y_i^T$, x_i, y_i are positive vectors with $y_i^T x_i = 1$, and s is the rank of B. The proof now follows by Lemmas 4, 5, and 6.

The converse is clear. ■

Proof of Theorem 2.

In the proof of Theorem 1, we observe that if B is symmetric, then $A_i = x_i x_i^T$, where x_i is a positive vector with $x_i^T x_i = 1$. This completes the proof. ■

4. APPLICATIONS OF MAIN RESULTS

In this section we use our main results to obtain characterizations of nonnegative matrices A such that A^k is 0-symmetric and $A^{k+1} = A$ for some positive integer k. This gives, in particular, characterization of matrices A whose generalized inverses are some power of A (cf. [1], [5]).

THEOREM 3. *Let A be a nonnegative matrix. Then A^m is 0-symmetric and $A^{m+1} = A$ if and only if there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following (not necessarily all) three* *types*:

(i) xy^T, *where* x *and* y *are positive vectors with* $y^Tx = 1$.

(ii)

$$\begin{bmatrix} 0 & w_{12}x_1 y_2^T & 0 & 0 & \cdots & & 0 \\ 0 & 0 & w_{23}x_2 y_3^T & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & w_{d-1d}x_{d-1} y_d^T \\ w_{d1}x_d y_1^T & 0 & 0 & \cdots & & 0 & 0 \end{bmatrix},$$

where x_i, y_i *are positive vectors of the same order with* $y_i^T x_i = 1$; x_i *and* x_j, $i \neq j$, *are not necessarily of the same order*; $d|m$; *and* $w_{12}, \dots, w_{d1}$ *are positive numbers with* $w_{12}w_{23}\cdots w_{d1} = 1$.

(iii) *A zero matrix*.

Proof. *"Only if" part*: Clearly A^m is idempotent. Hence by Theorem 1, there exists a permuation matrix P such that PAP^T is a direct sum of the square matrices of the types (I), (II), or (III). Since $A^{m+1} = A$, each summand S of PAP^T satisfies $S^{m+1} = S$. If S is of type (I), then $S = C_{11}$, where $C_{11}^m = xy^T$ for some positive vectors x and y such that $y^Tx = 1$. Since xy^T is idempotent of rank 1, there exists an invertible matrix U such that

$$xy^T = U\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U^{-1},$$

where zero block on the diagonal stands for a square matrix. It follows then that the first column of U is x, and the first row of U^{-1} is y^T. From Gantmacher [4, p. 235] we have

$$\sqrt[m]{xy^T} = U\begin{pmatrix} \sqrt[m]{1} & 0 \\ 0 & \sqrt[m]{0} \end{pmatrix}U^{-1}.$$

Further, if $R^{m+1} = R$ for some $R \in +\sqrt[m]{xy^T}$, then

$$R \in U\begin{pmatrix} \sqrt[m]{1} & 0 \\ 0 & 0 \end{pmatrix}U^{-1}.$$

Since $S \in +\sqrt[m]{xy^T}$, we obtain

$$S = U\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}U^{-1} = xy^T.$$

If S is of type (II), then

$$S = \begin{bmatrix} 0 & C_{12} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & C_{23} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{d-1d} \\ C_{d1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $(C_{12}C_{23}\cdots C_{d1})^{m/d} = x_1 y_1^T, \ldots, (C_{d1}C_{12}\cdots C_{d-1d})^{m/d} = x_d y_d^T$, x_i and y_i are positive vectors with $y_i^T x_i = 1$, $d|m$, and the zeros on the diagonal stand for the square matrices of appropriate orders. Therefore,

$$S^m = \begin{bmatrix} x_1 y_1^T & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & x_2 y_2^T & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & x_d y_d^T \end{bmatrix}.$$

Since S^m is an idempotent matrix of rank d, there exists an invertible matrix U such that

$$S^m = U\begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}U^{-1}.$$

This implies that the first d columns of U are

$$\begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_d \end{bmatrix}$$

and the first d rows of U^{-1} are $(v_1^T\ 0\ \cdots\ 0),\ldots,(0\ \cdots\ 0\ v_d^T)$ in this order, and I_d is the $d\times d$ identity matrix. Let

$$u_i=\begin{bmatrix} x_{i1}\\ x_{i2}\\ \vdots\\ x_{il_i}\end{bmatrix}\quad\text{and}\quad v_i=\begin{bmatrix} y_{i1}\\ y_{i2}\\ \vdots\\ y_{il_i}\end{bmatrix},\qquad 1\leqslant i\leqslant d.$$

From Gantmacher [4, p. 235]

$$S\in U\begin{bmatrix}\sqrt[m]{I_d} & 0\\ 0 & \sqrt[m]{0}\end{bmatrix}U^{-1}.$$

Since $S^{m+1}=S$, we get

$$S\in U\begin{pmatrix}\sqrt[m]{I_d} & 0\\ 0 & 0\end{pmatrix}U^{-1}.$$

Thus

$$S=U\begin{pmatrix}W & 0\\ 0 & 0\end{pmatrix}U^{-1},$$

where $W^m=I_d$. Also

$$S=\begin{bmatrix} 0 & C_{12} & 0 & 0 & \cdot & \cdot & \cdot & 0\\ 0 & 0 & C_{23} & 0 & \cdot & \cdot & \cdot & 0\\ \vdots & \vdots & \vdots & & & & & \vdots\\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & C_{d-1d}\\ C_{d1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0\end{bmatrix}.$$

So if $W=(w_{ij})$, then simple computations give all $w_{ij}=0$ except

$_{12}, w_{23}, \ldots, w_{d-1d}, w_{d1}$, and $w_{12}w_{23}\cdots w_{d1}=1$. Hence

$$S = U\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$$

$$= \begin{bmatrix} 0 & w_{12}x_1y_2^T & 0 & 0 & \cdots & & 0 \\ 0 & 0 & w_{23}x_2y_3^T & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & & \cdots & 0 & w_{d-1d}x_{d-1}y_d^T \\ w_{d1}x_dy_1^T & 0 & 0 & & \cdots & 0 & 0 \end{bmatrix}.$$

inally, suppose S is of type (III). Then $S^{m+1}=S$ gives $S=0$, completing the oof.

The converse is clear. ■

THEOREM 4. *Let A be a nonnegative matrix. Then* $A^{\dagger}=A^{m-1}$ *for some sitive integer m if and only if there exists a permutation matrix P such at* PAP^T *is a direct sum of matrices of the following (not necessarily all) ree types:*

(i) xx^T, *where x is a positive unit vector.*

(ii)

$$\begin{bmatrix} 0 & w_{12}x_1x_2^T & 0 & 0 & \cdots & & 0 \\ 0 & 0 & w_{23}x_2x_3^T & 0 & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & & \cdots & 0 & w_{d-1d}x_{d-1}x_d^T \\ w_{d1}x_dx_1^T & 0 & 0 & & \cdots & 0 & 0 \end{bmatrix},$$

here x_i *are positive unit vectors;* x_i *and* x_j, $i\neq j$, *are not necessarily of the me order;* $d|m$; *and* $w_{12},\ldots,w_{d1}$ *are positive numbers with* $w_{12}w_{23}\cdots w_{d1}$ 1.

(iii) *A zero matrix.*

Proof. Follows from Theorems 2 and 3. ■

5. REMARKS

(1) As special cases of Theorem 4 we can obtain theorems of Harary and Minc [5] and Berman [1], characterizing nonnegative matrices A such that $A^{-1}=A$ and $A^{\dagger}=A$ respectively.

(2) We can also derive the nonnegative solutions of the matrix equation $X^m=I$, where m is a positive integer, from Theorem 4. The solutions are square matrices A such that for some permutation matrix P, PAP^T is a direct sum of matrices A_i, where A_i is an identity matrix or a matrix of the form

$$\begin{bmatrix} 0 & a_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & a_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & a_{d-1} \\ a_d & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

where $a_1a_2\cdots a_{d-1}a_d=1$, $a_i>0$, $1\leqslant i\leqslant d$, and $d|m$.

The referee has informed us that M. Lewin [7] has also characterized the nonnegative solutions of $X^m=1$.

(3) A special case of Theorem 3 answers a question of Berman [1] for characterizing the nonnegative matrices which are equal to a (1,2)-inverse of themselves (equivalently $A=A^D$) under the hypothesis that A^2 is 0-symmetric. We note from Theorem 3 that if $A^3=A$, (i.e., A is equal to a (1,2)-inverse), then A is 0-symmetric if and only if A^2 is 0-symmetric.

(4) In another paper [6] we have characterized nonnegative matrices A whose Moore-Penrose generalized inverse $A^{\dagger}$ is nonnegative and is equal to some polynomial in A with scalar coefficients. This result generalizes Theorem 4 of this paper.

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