A note on the adjoint group of a ring*)

By

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Let R be a right artinian ring, N the Jacobson radical of R, and ${}^0R$ the adjoint group, that is, the group of quasi-regular elements of R.

We first prove

Theorem 1. *${}^0R$ is solvable if and only if*

$$R/N \cong \oplus \sum_{i=1}^{l} F^{(i)} \oplus \sum_{j=1}^{m} M_2^{(j)}(I/(2)) \oplus \sum_{k=1}^{n} M_2^{(k)}(I/(3)),$$

where $F^{(i)}$ are fields, $M_2^{(j)}(I/(2))$ are 2×2 matrix rings over integers modulo 2, and $M_2^{(k)}(I/(3))$ are 2×2 matrix rings over integers modulo 3.

Proof. Let ${}^0R$ be solvable. Write

$$R/N \cong \sum_{i=1}^{n} M_{n_i}(D_i),$$

where $M_{n_i}(D_i)$ are $n_i \times n_i$ matrix rings over division rings.

Since ${}^0(R/N) \cong {}^0R/{}^0N$, and the homomorphic image ${}^0R/{}^0N$ of ${}^0R$ is also solvable, we get by Hua [5] each D_i must be a field. Further if $n_i \geqq 2$ and the cardinality of D_i is greater than 3, then by using [6], p. 292, the group of quasi-regular elements of $M_{n_i}(D_i)$ cannot be solvable. Thus we get

$$R/N \cong \oplus \sum_{i=1}^{l} F^{(i)} \oplus \text{finite direct sum of matrix rings over fields having elements} \leqq 3,$$

where $F^{(i)}$ are fields. Also it is known that the group of quasi-regular elements in the 3×3 matrix ring over $I/(2)$ or $I/(3)$ is insoluble (Dickson [4], p. 309). Thus the group of quasi-regular elements of $M_{n_i}(I/(2))$ or $M_{n_i}(I/(3))$ is insoluble if $n_i \geqq 3$. This proves the theorem in one direction. The converse follows immediately since ${}^0R/{}^0N$ is clearly solvable, and N being a nilpotent ideal implies that ${}^0N$ is also solvable. As a special case we get

Corollary. *If ${}^0R$ is nilpotent then R/N is a finite direct sum of fields.*

This follows from the fact that the group of units of an $n \times n$ $(n > 1)$ matrix ring over a field cannot be nilpotent. The above corollary gives, in particular, theorem (a) of Bateman and Coleman [2].

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*) Presented in person at the annual conference of the Indian Mathematical society, 1968.

Theorem 2. *If ^0R is finitely generated and solvable then R is finite.*

Proof. Since ^0R is finitely generated and solvable we get that the fields in the expression for R/N in theorem 1 are finitely generated, and hence finite. Thus R/N is finite. This along with the fact that ^0R is finitely generated implies ^0N is finitely generated. Also ^0N is nilpotent, so that by WATTERS [7], the additive group N^+ is finitely generated, and hence N is finite because R is artinian. This completes the proof.

Corollary. *If ^0R is a finitely generated nilpotent group then $^0R = {^0N} \circ {^0A}$, where ^0A is abelian.*

The proof follows from the above theorems and also by using the facts that a finite ring is a direct sum of rings of prime orders and a finite nilpotent group is a direct product of sylow subgroups.

Theorem 3. *Let F be a field of characteristic p, and G a finite group. Then the group of units of the group ring $R = FG$ is nilpotent if and only if G is nilpotent and each q-sylow subgroup of G, $q \neq p$, is abelian.*

Proof. Let the group of units of R be nilpotent and H be a q-sylow subgroup of G, $q \neq p$. Then FH is semi-simple artinian and has the group of units nilpotent. Thus it immediately follows from corollary to theorem 1 that H is abelian. Conversely, by the hypothesis, we can write $G = P \times A$, where P is a p-sylow subgroup and A is an abelian subgroup whose order is not divisible by p. Then the group ring FA is commutative, semi-simple artinian, and, therefore, it is a finite direct sum of fields F_i which are algebras over F. Now

$$FG \cong FP \otimes FA \cong \oplus \sum (FP \otimes F_i) \cong \oplus \sum F_i P.$$

Thus the proof will be complete if we prove

Lemma. *Let K be a field of characteristic p and P be a finite p-group. Then the group U of units in the group ring $R = KP$ is nilpotent.*

Proof. It is well-known that the radical N of KP is a nilpotent ideal generated by $1 - g$, $g \in P$. Clearly each element of KP can then be put in the form $a - n$, $a \in K$ and $n \in N$ and is a unit if and only if $a \neq 0$. Thus $U = \{a - n \mid n \in N,\ 0 \neq a \in K\}$. Direct computation shows that if k is the degree of nilpotency of N, then

$$(u_1, u_2, \ldots, u_k) = 1,$$

for each u_i in U. Hence U is nilpotent.

Added after submitting the paper (received November 5, 1969):

Professor K. ELDRIDGE has given to one of the authors a copy of preprint of his paper "On ring structures determined by groups" (to be called "E" henceforth) which is about to appear in the Proc. Amer. Math. Soc. and in which similar results have been obtained. In the light of E we add the following remarks:

(1) Theorem 1 in E is incorrect. The solvable linear group $GL(2,3)$ provides a counter example. Our theorem 1 corrects the theorem in E. The cor. 2 to theorem 1 in E is then not correct but the cor. 1 remains true.

(2) Our theorem 2 gives a very short proof of his theorem 2: If R is artinian such that ${}^0R/N$ is torsion then R is finite. The proof of our theorem 2 also works if we assume ${}^0R/N$ is torsion instead of ${}^0R$ being solvable. Under this assumption we do not have to invoke theorem 1 to prove R/N is finite but just note that then the direct summands of R/N are matrix rings over commutative division rings, and since ${}^0R$ is finitely generated, R/N must be finite.

In fact the proof given by ELDRIDGE for the theorem 2 in E needs to be modified since it depends upon the prop. 2 which has a gap in its proof.

References

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Eingegangen am 5. 11. 1969

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