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NONSINGULAR SEMIPERFECT CS-RINGS II

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ABSTRACT. In this paper we obtain the precise structure of right nonsingular semiperfect right CS-rings. It is shown that such rings are direct sums of certain specific block matrix rings.

1. INTRODUCTION

Rings in which each closed right ideal is a direct summand have been studied by many authors. Such rings are called right CS-rings. Nonsingular right CS-rings have been considered by Chatters and Hajarnavis ([2]), Chatters and Khuri ([3]), Barthwal et. al. ([1]) and others. In this paper we continue the study of right nonsingular semiperfect right CS-rings and obtain their structure, in general (Theorem 3.4). As particular cases we obtain the structure theorems in [1] and [2].

2. NOTATION AND PRELIMINARIES

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right unital. For any two R-modules M and N, M is said to be N-injective if for every submodule K of N, any R-homomorphism $\phi : K \to M$ can be extended to an R-homomorphism $\widehat{\phi} : N \to M$. Equivalently, for any R-homomorphism $\phi : N \to E(M)$, where $E(M)$ is the injective hull of M, $\phi(N) \subset M$. M is said to be self-injective if it is M-injective. Self-injective modules are also commonly called quasi-injective modules. A submodule N of M is said to be essential in a submodule N^e of M with $N^e \supset N$, denoted by $N \subset_e N^e$, if for any nonzero submodule L of N^e, $N \cap L \neq 0$. Moreover, in this case N^e is called an essential extension of N in M. A submodule K of M is said to be essentially closed (or simply closed) in M if for any essential extension K^e of K in M, $K^e = K$. M is called a CS module if every submodule of M is essential in a direct summand of M, or equivalently, if every closed submodule of M is a direct summand of M. CS-modules have also been called extending modules (c.f. [4]) in

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the literature. If M has finite uniform dimension, then M is CS if and only if every uniform closed submodule of M is a direct summand of M ([4], Corollary 7.8).

A ring R is said to be a right CS-ring if the right module R_R is CS. R is semiperfect if it has a complete set $\{e_i\}_{i=1}^n$ of primitive orthogonal idempotents such that each e_iRe_i is a local ring. R is said to be right nonsingular if its right singular ideal $Z(R_R) = \{r \in R \mid rI = 0$ for some essential right ideal I of $R\}$ is zero. The term regular ring will mean von Neumann regular ring. R is said to be a right valuation ring if for any two right ideals I and J either $I \subset J$ or $J \subset I$. A left valuation ring is defined similary.

>From now on, unless otherwise stated, R will denote a right nonsingular semiperfect ring. Thus there exists a complete set $\{e_i\}_{i=1}^n$of orthogonal idempotents in R where each e_iRe_i is a local ring. In addition, $Q = Q^r_{max}(R)$ will denote right maximal quotient ring of R.

The following Lemmas are from [1]. We state them here without proof for convenience.

Lemma 2.1. *([1], Lemma 3.1) If R is a right CS-ring, then each e_iR is a uniform right ideal and e_iRe_i is a local domain.*

Lemma 2.2. *([1], Lemma 3.2) If R is right CS-ring and e_iR is not embeddable in e_jR, then e_iR is e_jR-injective.*

Corollary 2.3. *If R is indecomposable right CS-ring and e_iR is not embeddable in e_jR then $e_iRe_j = e_iQe_j$. In particular, $e_iRe_j \neq 0$ and so e_jR is embeddable in e_iR.*

Proof. By Lemma 2.2, e_iR is e_jR-injective. Thus for every $q \in Q$ and $r \in R$, $e_iqe_je_jr \in e_iR$. In particular, $e_iqe_j \in e_iR$ and hence $e_iRe_j = e_iQe_j$. Since R is indecomposable, $e_iQ \simeq e_jQ$ for all i and j. Thus e_iQe_j and hence e_iRe_j is nonzero. ∎

The proof of the next corollary is immediate from Corollary 2.3.

Corollary 2.4. *If R is an indecomposable right CS-ring then for $1 \leq i,j \leq n$ either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$.*

3. Structure of Semiperfect CS-Rings

In this section we obtain the structure of right nonsingular semiperfect right CS-rings. We begin with a simple observation.

Remark 3.1. *Suppose R is a right CS-ring. By Lemma 2.1 each e_iR is uniform and e_iRe_i is a local domain. Write* $R = (\bigoplus_{i\in I_1} e_iR) \oplus (\bigoplus_{i\in I_2} e_iR) \oplus$

$\ldots \oplus (\bigoplus_{i\in I_k} e_iR)$, *where for all* $i \in I_u$ *and* $j \in I_v$, $e_iQ \simeq e_jQ$ *if and only if* $u = v$. *It is not hard to see that* $R = (\bigoplus_{i\in I_1} e_iR) \oplus (\bigoplus_{i\in I_2} e_iR) \oplus \ldots \oplus (\bigoplus_{i\in I_k} e_iR)$ *is a ring decomposition of* R. *In view of this we will henceforth assume that the ring* R *is indecomposable. Thus,* Q *is a simple artinian ring and* $e_iQ \simeq e_jQ$ *for all* i *and* j.

Remark 3.2. *It is well known that the lattice* $L^*(Q)$ *of closed right ideals of* Q *is isomorphic to the lattice* $L^*(R)$ *of closed right ideals of* R *under the correspondence* $A \to A \cap R$. *Thus the uniform closed right ideals of* R *are precisely those of the form* $eQ \cap R$ *where* eQ *is any minimal right ideal of* Q. *If* $Q = M_n(D)$ *then any minimal, (equivalently uniform closed), right ideal* I *of* Q *is of the form*

$$I = \left\{ \begin{pmatrix} a_1x_1 & a_1x_2 & \ldots & a_1x_n \\ a_2x_1 & a_2x_2 & \ldots & a_2x_n \\ \ldots & \ldots & \ldots & \ldots \\ a_nx_1 & a_nx_2 & \ldots & a_nx_n \end{pmatrix} \mid x_j \in D,\ 1 \leq j \leq n \right\}$$

for some fixed $a_i \in D$, $1 \leq i \leq n$, *not all of which are zero. This fact will play a crucial role in the proof of our main structure theorem.*

As stated earlier there exists an independent family $\mathcal{F} = \{e_iR \mid 1 \leq i \leq n\}$ of indecomposable right ideals such that $R = \oplus\sum_{i=1}^{n} e_iR$. Renumbering, if necessary, we may write $R = [e_1R] \oplus [e_2R] \oplus \ldots \oplus [e_kR]$ where $[e_iR]$ denote the direct sum of those indecomposable right ideals in $\mathcal{F}$ which are isomorphic to e_iR. By Corollary 2.4, for $1 \leq i, j \leq k$ with $i \neq j$, either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$, that is, either $Hom(e_jR, e_iR) \neq 0$ or $Hom(e_iR, e_jR) \neq 0$. Among all diagrams of right ideals e_iR, $1 \leq i \leq k$ with nonzero R-homomorphisms, let $e_{i_1}R \to e_{i_2}R \to \ldots \to e_{i_r}R$ be a diagram with largest length r where $i_1, i_2, \ldots i_r$ are all distinct. We claim that $r = k$. If not, consider $e_{i_{r+1}}R \in \mathcal{F}\backslash\{e_{i_1}R, e_{i_2}R, \ldots, e_{i_r}R\}$. Clearly, $Hom(e_{i_r}R, e_{i_{r+1}}R) = 0$ and $Hom(e_{i_{r+1}}R, e_{i_1}R) = 0$. By Corollary 2.4, $Hom(e_{i_1}R, e_{i_{r+1}}R) \neq 0$ and $Hom(e_{i_{r+1}}R, e_{i_r}R) \neq 0$. Consequently, we get the diagram

$$\begin{array}{ccccccc} e_{i_1}R & \to & e_{i_2}R & \to & \ldots & \to & e_{i_r}R \\ & \searrow & & & & & \\ & & e_{i_{r+1}}R & & & & \end{array}$$

with nonzero maps. Now, if $Hom(e_{i_{r+1}}R, e_{i_2}R) \neq 0$, then we get a longer diagram

$$e_{i_1}R \to e_{i_{r+1}}R \to e_{i_2}R \to \ldots \to e_{i_r}R$$

with nonzero maps, a contradiction. Thus $Hom(e_{i_{r+1}}R, e_{i_2}R) = 0$. Then $Hom(e_{i_2}R, e_{i_{r+1}}R) \neq 0$ and we continue in a similar manner to obtain $Hom(e_{i_{r+1}}R, e_{i_r}R) = 0$, a contradiction. Thus $r = k$. Renumbering again, if necessary, we can assume that the largest diagram with nonzero maps is

$$e_kR \rightarrow e_{k-1}R \rightarrow ... \rightarrow e_1R.$$

Thus for all i, j with $1 \leq i \leq j \leq k$, $e_iRe_j \neq 0$. Under this ordering, if for some $i > j$, $e_iRe_j = 0$ then $e_lRe_m = 0$ for all $l \geq i$ and $m \leq j$. To see this, let $e_lRe_m \neq 0$ for some $l \geq i$ and $m \leq j$. Since e_iR's are uniform, $e_iRe_l \neq 0$, and $e_mRe_j \neq 0$ we get $(e_iRe_l)(e_lRe_m)(e_mRe_j) \neq 0$, that is, $e_iRe_j \neq 0$, a contradiction. Further since $e_mRe_l = e_mQe_l$ and $e_uQ \simeq e_vQ$ for all u and v, it follows that for all $l \geq i$ and $m \leq j$, $e_mRe_l \simeq e_kQe_k$ (as additive groups).

Theorem 3.1. *Suppose R is an indecomposable right nonsingular semi-perfect right CS-ring. Then there exists a division ring D and positive integers $n_1, n_2, ..., n_k$ such that*

$$R \simeq \begin{pmatrix} M_{n_1}(D_{11}) & M_{n_1 \times n_2}(D_{12}) & \cdots \cdots & M_{n_1 \times n_k}(D_{1k}) \\ M_{n_2 \times n_1}(D_{21}) & M_{n_2}(D_{22}) & \cdots \cdots & M_{n_2 \times n_k}(D_{2k}) \\ \cdots & \cdots & \cdots \cdots & \cdots \\ M_{n_{k-1} \times n_1}(D_{k-11}) & M_{n_{k-1} \times n_2}(D_{k-12}) & \cdots \cdots & M_{n_{k-1} \times n_k}(D_{k-1k}) \\ M_{n_k \times n_1}(D_{k1}) & M_{n_k \times n_2}(D_{k2}) & \cdots \cdots & M_{n_k}(D_{kk}) \end{pmatrix}$$

where for each $i, 1 \leq i \leq k$, D_{ii} is a local domain contained in D, for i, j with $1 \leq i, j \leq k$, D_{ij} is an additive subgroup of D such that $D_{ij}D_{jl} \subset D_{il}$ for $1 \leq l \leq k$. Furthermore, (i) for $1 \leq i \leq j \leq k$, $D_{ij} \neq 0$, (ii) for $i > j$ if $D_{ij} = 0$ then for all $l \geq i$ and $m \leq j$, $D_{lm} = 0$ and $D_{ml} = D$, (iii) if for any i, $n_i > 1$ then for every $c \in D$ either $c \in D_{ii}$ or $c^{-1} \in D_{ii}$, or equivalently, $M_{n_i}(D_{ii})$ is a right CS-ring (in this case D_{ii} is, indeed, a right and left valuation domain having D as its right and left classical quotient ring, (iv) if for $1 \leq i, j \leq k$ with $i \neq j$, D_{ij} and D_{ji} are both nonzero then for every $c \in D$ either $c \in D_{ij}$ or $c^{-1} \in D_{ji}$, (v) for $1 \leq i \leq k$, the injective hull of D_{ik} as right D_{kk}-module is D, in particular, D_{kk} is always a right Ore domain with D as its right classical quotient ring.

Proof. With the notation preceding this theorem, $R = [e_1R] \oplus [e_2R] \oplus ... \oplus [e_kR]$ where for all i, j with $1 \leq i < j \leq k$, $e_iRe_j \neq 0$. For $1 \leq i \leq k$, let n_i denote the number of direct summands in $[e_iR]$. Then $R \simeq \left(M_{n_i \times n_j}(e_iRe_j)\right)$, a $k \times k$ block matrix ring. Let $D = e_kQe_k$. D is a division ring. Let $0 \neq q \in Q$ and for $1 \leq i \leq k$, let $\theta_i = e_kqe_i$.

Because $e_iQ \simeq e_jQ$, there exists a unique $e_iq_ie_k \in e_iQe_k$ such that $(e_iq_ie_k)(e_kqe_i) = e_i$ and $(e_kqe_i)(e_iq_ie_k) = e_k$. For simplicity we set $\theta_i^{-1} = e_iq_ie_k$. Let $D_{ij} = \theta_ie_iRe_j\theta_j^{-1} \subset D$. Since for $1 \leq i,j,l \leq k$, $(e_iRe_j)(e_jRe_l) \subset e_iRe_l$, it follows that $D_{ij}D_{jl} \subset D_{il}$. Also, for $1 \leq i \leq k$, D_{ii} is a local domain contained in D, for $1 \leq i,j \leq k$ with $i \neq j$, D_{ij} is an additive subgroup of D and

$$R \simeq \begin{pmatrix} M_{n_1}(D_{11}) & M_{n_1\times n_2}(D_{12}) & \cdots & \cdots & M_{n_1\times n_k}(D_{1k}) \\ M_{n_2\times n_1}(D_{21}) & M_{n_2}(D_{22}) & \cdots & \cdots & M_{n_2\times n_k}(D_{2k}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{n_{k-1}\times n_1}(D_{k-11}) & M_{n_{k-1}\times n_2}(D_{k-12}) & \cdots & \cdots & M_{n_{k-1}\times n_k}(D_{k-1k}) \\ M_{n_k\times n_1}(D_{k1}) & M_{n_k\times n_2}(D_{k2}) & \cdots & \cdots & M_{n_k}(D_{kk}) \end{pmatrix}.$$

Now if for $1 \leq i,j \leq k$, $D_{ij} = 0$ then $e_iRe_j = 0$, i.e., e_jR is not embeddable in e_iR. Consequently, by Corollary 2.3, $e_jRe_i = e_jQe_i$. Now for every $q \in Q$, $e_kqe_k = \theta_j\theta_j^{-1}e_kqe_k\theta_i\theta_i^{-1}$. Since $\theta_j^{-1}e_kqe_k\theta_i \in e_jQe_i = e_jRe_i$, it follows that $e_kqe_k \in D_{ji}$. Thus $D_{ji} = D$. Moreover, under the ordering described above for $1 \leq i < j \leq k$, $D_{ij} \neq 0$ and if for some $i > j$, $D_{ij} = 0$ and $D_{i_1j} \neq 0$ then for all $l \geq i$ and $m < i$, $D_{lm} = 0$ and hence $D_{ml} = D$.

We will now show that if for some i,j, $i \neq j$, both D_{ij} and D_{ji} are nonzero then for every $c \in D$ either $c \in D_{ij}$ or $c^{-1} \in D_{ji}$. So, let both D_{ij} and D_{ji} be nonzero where $i \neq j$. To be definite, let $i > j$. Let, if possible, there exist $c \in D$ such that neither $c \in D_{ij}$ nor $c^{-1} \in D_{ji}$. Let $t_1 = \sum_{r=1}^{j} n_r$, $t_2 = \sum_{r=1}^{i-1} n_r + 1$, and let

$$U = \left\{ \sum_{i=1}^{n} a_ie_{t_1i} + \sum_{i=1}^{n} ca_ie_{t_2i} \mid a_i \in D,\ 1 \leq i \leq n \right\}.$$

Then U is a minimal right ideal of Q (or equivalently, a uniform closed right ideal of Q). It follows that $U \cap R$ is a uniform closed right ideal of R. We will show that U does not contain an idempotent in R. If $x = \sum_{i=1}^{n} a_ie_{t_1i} + \sum_{i=1}^{n} ca_ie_{t_2i}$ is an idempotent in $U \cap R$, then at least one of a_{t_1} and a_{t_2} is nonzero, $a_{t_1} \in D_{jj}$, $ca_{t_1} \in D_{ij}$, $a_{t_2} \in D_{ji}$, $ca_{t_2} \in D_{ii}$. Since $x^2 = x$, $a_{t_1}^2 + a_{t_2}ca_{t_1} = a_{t_1}$. First assume that $a_{t_1} \neq 0$. If a_{t_1} is invertible in D_{jj} then $ca_{t_1} \in D_{ij}$ will yield $c \in D_{ij}D_{jj} \subset D_{ij}$, a contradiction. Therefore, let a_{t_1} be not invertible in D_{jj}. Since $a_{t_1} \in D_{jj}$ and D_{jj} is a local domain, $1 - a_{t_1}$ is invertible in D_{jj}. Since $a_{t_1}^2 + a_{t_2}ca_{t_1} = a_{t_1}$, we have $a_{t_1} + a_{t_2}c = 1$ (in D). Thus $c^{-1} = (1 - a_{t_1})^{-1}a_{t_2} \in D_{jj}D_{ji} \subset D_{ji}$, a contradiction. Now, let $a_{t_1} = 0$. Then, $a_{t_2} \neq 0$. Using once again

$x^2 = x$ we get $a_{t_2} c a_{t_2} = a_{t_2}$. Consequently, $a_{t_2} c = c a_{t_2} = 1$ (in D). But then $c^{-1} = a_{t_2} \in D_{ji}$, once again a contradiction to the choice of c.

Similarly it can be proved that if for some i, $n_i > 1$ then for each $c \in D$, either $c \in D_{ii}$ or $c^{-1} \in D_{ii}$. In particular, D_{ii} is a right and left Ore (indeed valuation) domain with D as its right and left classical quotient ring. The ring $M_{n_i}(D_{ii})$ is a right CS-ring follows from ([1], Lemma 3.6).

Let R_1 denote the block matrix ring $(M_{n_i \times n_j}(D_{ij}))$. Note that $Q^r_{max}(R_1) = M_n(D)$ where $n = \sum_{i=1}^{k} n_i$. We will now show that D_{kk} is always a right Ore domain with D as its right maximal quotient ring. Since D_{kk} is nonsingular, it is sufficient to show that $D_{kk} \subset_e D$ as right D_{kk}-module. So, let $x \in D$ and consider $y = x e_{nn} \in M_n(D)$. Since $R_1 \subset_e M_n(D)$, there exists $z = \sum_{i,j=1}^{n} a_{ij} e_{ij}$ such that $0 \neq yz \in R_1$, i.e., $0 \neq \sum_{j=1}^{n} x a_{nj} e_{nj} \in R_1$. If $x a_{nn} \neq 0$ then we are done. So, let $x a_{nn} = 0$. Let j_0 be such that $x a_{nj_0} \neq 0$. Let $\sum_{r=1}^{i-1} n_r + 1 \leq j_0 \leq \sum_{r=1}^{i} n_r$. Since $D_{ik} \neq 0$, choose $0 \neq a \in D_{ik}$ and consider $0 \neq a e_{j_0 n} \in R_1$. Since $\left(\sum_{j=1}^{n} x a_{nj} e_{nj}\right) a e_{j_0 n} = x a_{nj_0} a e_{nn} \in R_1$, it follows that $0 \neq x a_{nj_0} a \in D_{kk}$. Observe that $a_{nj_0} a \in D_{ki} D_{ik} \subset D_{kk}$. Thus, D_{kk} is essential in D, as desired. It can be similarly shown that for $1 \leq i \leq k$, $D_{ik} \subset_e D$ as a right D_{kk}-module. Since D is injective as right D_{kk}-module it follows that the injective hull of D_{ik} as a right D_{kk}-module is D. This completes the proof. ∎

Remark 3.3. *In Theorem 3.1, the condition: for every $c \in D$ either $c \in D_{ii}$ or $c^{-1} \in D_{ii}$ need not be satisfied when $n_i = 1$. For example, the ring $R = \begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ is right nonsingular right artinian right CS-ring.*

Theorem 3.2. *Suppose D is a division ring and $n_1, n_2, \ldots, n_k$ are positive integers. Let*

$$R = \begin{pmatrix} M_{n_1}(D_{11}) & M_{n_1 \times n_2}(D_{12}) & \cdots & \cdots & M_{n_1 \times n_k}(D_{1k}) \\ M_{n_2 \times n_1}(D_{21}) & M_{n_2}(D_{22}) & \cdots & \cdots & M_{n_2 \times n_k}(D_{2k}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{n_{k-1} \times n_1}(D_{k-11}) & M_{n_{k-1} \times n_2}(D_{k-12}) & \cdots & \cdots & M_{n_{k-1} \times n_k}(D_{k-1k}) \\ M_{n_k \times n_1}(D_{k1}) & M_{n_k \times n_2}(D_{k2}) & \cdots & \cdots & M_{n_k}(D_{kk}) \end{pmatrix}$$

where for $1 \le i \le k$, D_{ii} *is a local domain contained in* D, *for* $1 \le i,j \le k$, $i \ne j$, D_{ij} *is an additive subgroup of* D *such that for* $1 \le i,j,l \le k$, $D_{ij}D_{jl} \subset D_{il}$. *Assume that*

(i) for $1 \le i \le j \le k$, $D_{ij} \ne 0$,

(ii) for $i > j$, *if* $D_{ij} = 0$ *then* $D_{ji} = D$,

(iii) for $1 \le i \le k$, *the injective hull of* D_{ik} *as a right* D_{kk}*-module is* D,

(iv) if for some i, $n_i > 1$ *then for every* $c \in D$ *either* $c \in D_{ii}$ *or* $c^{-1} \in D_{ii}$, *(or equivalently* $M_{n_i}(D_{ii})$ *is a right CS-ring),*

(v) for $1 \le i,j \le k$, $i \ne j$, *for every* $c \in D$ *either* $c \in D_{ij}$ *or* $c^{-1} \in D_{ji}$.

Then R *is an indecomposable right nonsingular semiperfect right CS-ring.*

Proof. Clearly, R is an indecomposable right nonsingular semiperfect ring. We will show that R is a right CS-ring. Since R has finite uniform dimension, it is sufficient to show that every uniform closed right ideal of R contains an idempotent ([4], Corollary 7.8). Clearly, $Q = Q^r_{max}(R) = M_n(D)$, where $n = \sum_{j=1}^{k} n_j$. Let U be a minimal (equivalently, uniform closed) right ideal of Q. Then

$$U = \left\{ \begin{pmatrix} a_1x_1 & a_1x_2 & \dots & a_1x_n \\ a_2x_1 & a_2x_2 & \dots & a_2x_n \\ \dots & \dots & \dots & \dots \\ a_nx_1 & a_nx_2 & \dots & a_nx_n \end{pmatrix} \mid \text{for } 1 \le i \le n,\ x_i \in D \right\}$$

for some fixed $a_i \in D$, $(1 \le i \le n)$ not all of which are zero. We will show that U contains an idempotent of R. To do this, we will produce an element in U such that all columns but one are zero, the nonzero column has 1 on the main diagonal, and all other entries on the nonzero column belong to the appropriate D_{ij}' s. Clearly such an element lies in R and is an idempotent element.

Let p be the largest integer such that $a_p \ne 0$. For $1 \le i \le k$, let $N_i = \sum_{j=1}^{i} n_j$, $N_0 = 0$. Write $p = N_{i-1} + r$ where $1 \le r \le n_i$ and notice that both i and r are uniquely determined. If for every j such that $1 \le j \le i$, $a_l a_p^{-1} \in D_{ji}$ whenever $l = N_{j-1} + l_1$ for some l_1 with $1 \le l_1 \le n_j$, i.e., if for every l with $1 \le l \le p$, $a_l a_p^{-1}$ belong to the appropriate D_{ji}. Then taking $x_i = 0$ for $i \ne p$ and $x_p = a_p^{-1}$ we get an idempotent in $U \cap R$. Otherwise choose the largest integer p_1 such that $a_{p_1} a_p^{-1}$ does not belong to the appropriate D_{ji}. Notice that $p_1 < p$. Let $p_1 = N_{i_1} + r_1$ where $1 \le r_1 \le n_{i_1}$ so that by our assumption

$a_{p_1}a_p^{-1} \notin D_{i_1 i}$. Since $p_1 < p$, $i_1 \leq i$. If $i_1 = i$ then $n_i > 1$ and hence by condition (iii) of the hypothesis $a_p a_{p_1}^{-1} \in D_{ii_1} = D_{ii}$. If $i_1 \neq i$, then by condition (iv) of the hypothesis $a_p a_{p_1}^{-1} \in D_{ii_1}$. Thus, in either case, $a_p a_{p_1}^{-1} \in D_{ii_1}$. By our choice of p_1 for $p_1 < l \leq p$, $l = N_{j-1} + l_1$, $1 \leq l_1 \leq n_j$, $a_l a_p^{-1} \in D_{ji}$, that is, for $p_1 < l \leq p$, $a_l a_p^{-1}$ lie in the appropriate D_{ji}. Since $a_l a_{p_1}^{-1} = (a_l a_p^{-1})(a_p a_{p_1}^{-1})$ and $D_{ji} D_{ii_1} \subset D_{ji_1}$, it follows that for $p_1 < l \leq p$, $l = N_{j-1} + l_1$, $1 \leq l_1 \leq n_j$, $i_1 \leq j \leq i$, $a_l a_{p_1}^{-1} \in D_{ji_1}$. Clearly, $a_l a_{p_1}^{-1} \in D_{i_1 i_1}$ for $l = p_1$. Now, if for every j such that $1 \leq j \leq i$, $a_l a_{p_1}^{-1} \in D_{ji}$ whenever $l = N_{j-1} + l_1$ for some l_1 with $1 \leq l_1 \leq n_j$ then taking $x_i = 0$ for $i \neq p_1$ and $x_{p_1} = a_{p_1}^{-1}$ we get an idempotent in $U \cap R$. If not choose again the largest integer p_2 such that $a_{p_2} a_{p_1}^{-1}$ does not belong to the appropriate D_{ji_1} and continue as above. After a finite number of steps, we get an element in $U \cap R$ as described above. Thus $U \cap R$ contains an idempotent and hence is a summand. This completes the proof. ∎

Remark 3.4. *The following diagrammatic representation should help the reader to follow the scheme used in the proof of Theorem 3.2 to create an idempotent in the ideal $U \cap R$.*

$$\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \uparrow & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a_{p_2}a_{p_2}^{-1} = 1 \in D_{i_2 i_2} & \cdots & a_{p_2}a_{p_1}^{-1} \notin D_{i_2 i_1} & \cdots & \cdots & \cdots \\
\cdots & \uparrow & \swarrow & \uparrow & \cdots & \cdots & \cdots \\
\cdots & a_{p_1}a_{p_2}^{-1} \in D_{i_1 i_2} & \cdots & a_{p_1}a_{p_1}^{-1} = 1 \in D_{i_1 i_1} & \cdots & a_{p_1}a_p^{-1} \notin D_{i_1 i} & \cdots \\
\cdots & \cdots & \cdots & \uparrow & \swarrow & \uparrow & \cdots \\
\cdots & a_p a_{p_2}^{-1} \in D_{ii_2} & \cdots & a_p a_{p_1}^{-1} \in D_{ii_1} & \cdots & a_p a_p^{-1} = 1 \in D_{ii} & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots
\end{pmatrix}$$

We start by choosing $x_i = 0$ for $i > p$. Take $x_p = a_p^{-1}$. The upward arrow indicates that we move upwards in that column until we hit the first element which is not in the appropriate block (say this element is in the p_1-th row). The diagonal arrow at this point indicates we move to the diagonally opposite entry (in the p_1-th column). At this stage, we repeat the process by choosing $x_i = 0$ for $i > p_1$ and $x_{p_1} = a_{p_1}^{-1}$. Observe that the first element in this column which is not in the appropriate block is always in a row strictly above the p_1-th row. The process continues until we obtain the desired column all of whose entries are in the appropriate blocks. At this point we make the remaining *columns on the left (if any) zero by choosing the corresponding* x_i*' s as zeros.*

Lemma 3.3. *Under the hypothesis of Theorem 3.2, R is prime right Goldie if and only if each* $D_{ij} \neq 0$.

Proof. First assume that $D_{ij} \neq 0$ for all i and j with $1 \leq i, j \leq k$. We will prove that R is prime right Goldie. It is enough to show that R is semiprime. To see this, let N be a nilpotent ideal of R. Let f_i be the idempotent in R such that $f_i R f_i = M_{n_i}(D_{ii})$. Then $f_i N f_i$ is a nilpotent ideal of the prime ring $M_{n_i}(D_{ii})$ and so $f_i N f_i = 0$. Thus, N has zero blocks on the diagonal. In case some (i, j) entry of N is nonzero then a suitable multiplication of a nonzero element in R with N will produce an element in N with nonzero entry on the diagonal, a contradiction. Thus $N = 0$. It follows that R is semiprime right Goldie and hence prime right Goldie because $Q^r_{max}(R)$ is simple artinian. Indeed, $Q^r_{cl}(R) = Q^r_{max}(R)$.

To prove the converse, assume that $D_{ij} = 0$ for some $i > j$. Then $D_{lm} = 0$ for all $l \geq i$ and $m \leq j$, in particular $D_{i1} = 0$. Choose the smallest positive integer l_1 such that $D_{l_1 1} = 0$. Then $D_{11}, D_{21}, \ldots, D_{l_1-1,1}$ are all nonzero. On the other hand, for all j with $1 \leq j \leq l_1 - 1$, $D_{l_1 j} D_{j1} \subset D_{l_1 1} = 0$. It follows that $D_{l_1 j} = 0$ for all j with $1 \leq j \leq l_1 - 1$. Also then for all $l \geq l_1$ and $1 \leq j \leq l_1 - 1$, $D_{lj} = 0$ and $D_{jl} = D$. Let

$$S_{11} = \begin{pmatrix} M_{n_1}(D_{11}) & M_{n_1 \times n_2}(D_{12}) & \ldots & M_{n_1 \times n_{l_1-1}}(D_{1,l_1-1}) \\ M_{n_2 \times n_1}(D_{21}) & M_{n_2}(D_{22}) & \ldots & M_{n_2 \times n_{k_i}}(D_{2,l_1-1}) \\ \ldots & \ldots & \ldots & \ldots \\ M_{n_{l_1-1} \times n_1}(D_{l_1-11}) & M_{n_{l_1-1} \times n_2}(D_{l_1-12}) & \ldots & M_{n_{l_1-1}}(D_{l_1-1,l_1-1}) \end{pmatrix}, \qquad \ldots(1)$$

$u_1 = \sum_{i=1}^{l_1-1} n_i$ and $v_1 = \sum_{i=l_1}^{k} n_i$. Then R becomes

$$\begin{pmatrix} S_{11} & M_{u_1 \times v_1}(D) \\ 0 & R' \end{pmatrix}.$$

Consequently, R is not prime. This completes our proof. ∎

Theorem 3.4. *For a ring R, the following are equivalent:*

(i) R is an indecomposable right nonsingular semiperfect right CS-ring.

(ii) R is isomorphic to the block matrix ring $(M_{n_i \times n_j}(D_{ij}))$ as in Theorem 3.2.

(iii) R is isomorphic to

$$\begin{pmatrix} S_{11} & S_{12} & \dots & S_{1l-1} & S_{1l} \\ 0 & S_{22} & \dots & S_{2l-1} & S_{2l} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_{l-1,l-1} & S_{l-1,l} \\ 0 & 0 & \dots & 0 & S_{ll} \end{pmatrix}$$

where for each i, S_{ii} is a prime right nonsingular semiperfect right CS-ring as in Lemma 3.3 and for $i < j$, S_{ij} consists of rectangular matrices over D of the appropriate size.

Proof. The implication $(i) \Rightarrow (ii)$ is Theorem 3.1, and $(ii) \Rightarrow (i)$ is Theorem 3.2. In light of Lemma 3.3 , it is clear that $(iii) \Rightarrow (ii)$. To prove $(ii) \Rightarrow (iii)$, start out proceeding as in the proof of Lemma 3.3 to obtain the representation of R as

$$\begin{pmatrix} S_{11} & M_{u_1 \times v_1}(D) \\ 0 & R' \end{pmatrix}.$$

Continue on recursively on R' to obtain the desired representation. ∎

Remark 3.5. *Using the structure theorem above, it follows that the classical ring of right quotients $Q^r_{cl}(R)$ of a semiperfect right nonsingular right CS-ring R is of the form*

$$\begin{pmatrix} M_{l_1}(D) & M_{l_1 \times l_2}(D) & \dots & M_{l_1 \times l_{r-1}}(D) & M_{l_1 \times l_r}(D) \\ 0 & M_{l_2}(D) & \dots & M_{l_2 \times l_{r-1}}(D) & M_{l_2 \times l_r}(D) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{l_{r-1}}(D) & M_{l_{r-1} \times l_r}(D) \\ 0 & 0 & \dots & 0 & M_{l_r}(D) \end{pmatrix}$$

and is, therefore, semiprimary. It follows, then, a posteriori, that the structure theorem ([2], Theorem 6.10) applies in this case. However, it does not seem possible to prove that $Q^r_{cl}(R)$ is semiprimary without using the structure of R in Theorem 3.4. In addtion, Theorem 3.4 also gives an intrinsic description of the diagonal blocks in the representation of R whereas these could be characterized only upto orders in simple artinian rings in ([2], Theorem 6.10).

Following the notation in [1], we will refer to the following condition as condition $(*)$:

If $e_iR \not\cong e_jR$, and if both e_iRe_j and e_jRe_i are nonzero then there exists $c : e_iQ \to e_jQ$ such that $c(e_iR) \not\subseteq e_jR$ and $c^{-1}(e_jR) \not\subseteq e_iR$.

Lemma 3.5. *Suppose R is an indecomposable right nonsingular semiperfect right CS-ring. Then R satisfies condition $(*)$ if and only if for*

$1 \leq i, j \leq n$ with $e_iR \not\cong e_jR$ either $e_iRe_j = 0$, $e_jRe_i \neq 0$ or $e_iRe_j \neq 0$, $e_jRe_i = 0$.

Proof. First assume R satisfies the condition $(*)$. Let $e_iR \not\cong e_jR$. Then, either $e_iRe_j = 0$ or $e_jRe_i = 0$ ([1], Lemma 3.4). By Corollary 2.4, either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$. Thus either $e_iRe_j = 0$, $e_jRe_i \neq 0$ or $e_iRe_j \neq 0$, $e_jRe_i = 0$. The converse is vacuously true. ∎

Suppose R is an indecomposable right nonsingular semiperfect right CS-ring satisfying the condition $(*)$. Then, by Lemma 3.5, for $1 \leq i, j \leq n$ with $e_iR \not\cong e_jR$ either $e_iRe_j = 0$, $e_jRe_i \neq 0$ or $e_iRe_j \neq 0$, $e_jRe_i = 0$. With the notations introduced in the proof of the Theorem 3.1, for $1 \leq i, j \leq n$ with $i \neq j$ either $D_{ij} = 0, D_{ji} \neq 0$ or $D_{ij} \neq 0, D_{ji} = 0$. Under the ordering described above $D_{ij} = 0$ for $i > j$. Thus, $D_{ij} = D$ for $i < j$. We thus have the following:

Corollary 3.6. ([1], *Theorem 3.9) Suppose R is an indecomposable right nonsingular semiperfect ring satisfying the condition $(*)$. Then R is a right CS-ring if and only if there exists a division ring D and positive integers $n_1, n_2, ..., n_k$ such that*

$$R \simeq \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \ldots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \ldots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \ldots & 0 & M_{n_k}(D_k) \end{pmatrix}$$

where for each i, D_i is a local domain contained in D, $Q(D_k) = D$. Furthermore, if for any i, $n_i > 1$, then (i) D_i is the right and left Ore domain with classical quotient ring D, (ii) for each $c \in D$, either $c \in D_i$ or $c^{-1} \in D_i$ and (iii) $M_{n_i}(D_i)$ is a right CS-ring.

Corollary 3.7. ([2], *Theorem 3.1) Suppose R is an indecomposable right nonsingular semiprimary ring. Then R is a right CS-ring if and only if there exists a division ring D and positive integers $n_1, n_2, ..., n_k$ such that*

$$R \simeq \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \ldots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \ldots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \ldots & 0 & M_{n_k}(D_k) \end{pmatrix}$$

where for each i, $1 \leq i \leq k$, D_i is a division subring of D. Furthermore, if for some i, $n_i > 1$ or $i = k$ then $D_i = D$.

Proof. First assume that R is a right CS-ring. Since $J(R)$, the Jacobson radical of R is nilpotent, for each i, $1 \leq i \leq k$, e_iRe_i is a division ring. Thus, with the notations of Theorem 3.1, each D_{ii} is a division ring. Also then, for $1 \leq i, j \leq k$, $i \neq j$, $e_iRe_j = 0$ or $e_jRe_i = 0$. Thus with the ordering of e_iR' s as in Theorem 3.1, $D_{ij} = 0$ for $i > j$ and $D_{ij} = D$ for $i < j$. Writing $D_i = D_{ii}$ for $1 \leq i \leq k$, we have the result.The last statement follows from the fact that D_{ii} is a division ring and the condition (ii) in Theorem 3.1. The converse follows from Theorem 3.2. ∎

We conclude by pointing out that right nonsingular semiperfectright CS-rings which do not satisfy the condition (*) are rather abundant. Indeed any prime semiperfect right CS-ring which is not a full matrix ring over a valuation domain does not satisfy (*). These rings are characterized in Lemma 3.3 above. A specific instance would be, for example,

$$\begin{pmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{pmatrix}$$

where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.

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