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Right-left symmetry of $aR \oplus bR = (a+b)R$ in regular rings

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Abstract

A somewhat surprising and unexpected result in the theory of von Neumann regular rings is proved. © 1998 Elsevier Science B.V. All rights reserved.

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Theorem 1. *Let R be a ring and let $a, b \in R$ such that $a+b$ is a von Neumann regular element. Then the following are equivalent:*

(i) $aR \oplus bR = (a+b)R$.
(ii) $Ra \oplus Rb = R(a+b)$.
(iii) $aR \cap bR = (0)$ *and* $Ra \cap Rb = (0)$.

Proof. (i) $\Rightarrow$ (ii). By hypothesis $(a+b)h(a+b) = (a+b)$, for some $h \in R$. Since $a, b \in (a+b)R$, we have that $a = (a+b)x$ and $b = (a+b)y$ for some $x, y \in R$. Then $(a+b)ha = (a+b)h(a+b)x = (a+b)x = a$. Similarly, $(a+b)hb = b$. Using $aR \cap bR = (0)$, we get

$$aha = a \quad bha = 0, \tag{1}$$

$$bhb = b \quad ahb = 0. \tag{2}$$

(1) and (2) yield $ah(a+b) = a$ and $bh(a+b) = b$. This proves

$$Ra + Rb = R(a+b).$$

To prove directness of the sum, let $ua = vb$ for some $u, v \in R$. This gives $uaha = vbha$ and so by (1), $ua = 0$ proving directness.

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(ii) $\Rightarrow$ (i) is symmetrical.

We shall now prove that (iii) $\Rightarrow$ (i). This will complete the proof of the theorem, because (i) or (ii) $\Rightarrow$ (iii) is trivial. Since $a+b$ is von Neumann regular in R, there exists some h in R such that $(a+b)h(a+b)=a+b$. From (iii), we have that $Ra \cap Rb=(0)$ and therefore,

$$(a+b)ha=a \quad (a+b)hb=b.$$

So $a,b \in (a+b)R$ and $aR+bR=(a+b)R$. Again from (iii) we have that $aR \cap bR=(0)$ and hence $aR \oplus bR=(a+b)R$. □

Remark 1. The question may be asked whether the theorem can be extended to possibly rectangular matrices over a von Neumann regular ring S. The answer is in the affirmative. The statements (i)–(iii) in the theorem will then read

(i)′ $a\Gamma \oplus b\Gamma=(a+b)\Gamma$,

(ii)′ $\Gamma a \oplus \Gamma b=\Gamma(a+b)$,

(iii)′ $a\Gamma \cap b\Gamma=(0)$ and $\Gamma a \cap \Gamma b=(0)$.

where a,b are $m \times n$ matrices over S such that there exists an $n \times m$ matrix x with $(a+b)x(a+b)=(a+b)$ and Γ is the additive group of all $n \times m$ matrices over S.

Remark 2. The statements (i)–(iii) in the theorem are related to a partial ordering '$\leq$' in a von Neumann regular ring R: For $a,b \in R$, $a \leq b$ if $ax=bx$ and $xa=xb$ for some x satisfying $axa=a$. It can be shown that each of the statements in the theorem is equivalent to

(iv) $a \leq a+b$.

Partial ordering '$\leq$' and applications of the above theorem to shorted operators in electrical networks as studied by Anderson [1] and Anderson–Trapp [2] will appear elsewhere.

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