

Nonsingular Semiperfect CS-Rings

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In this paper the structure of right nonsingular semiperfect right CS-rings has been considered. Under certain conditions these rings are shown to be direct sums of triangular blocked matrix rings over Ore domains. As a by-product it follows that the $n \times n$ matrix ring over a local domain D , $n > 1$, is a right CS-ring if and only if D is a right and left valuation domain. © 1998 Academic Press

1. INTRODUCTION

The purpose of this paper is to study right nonsingular semiperfect rings in which each closed right ideal is a direct summand. A ring in which each closed right ideal is a direct summand is called a right CS-ring. CS-modules and rings have been of considerable interest to many authors including Harada, Huynh, Oshiro, Osofsky, Smith, and Wisbauer (cf. [2–4]). In this paper it is shown that under a certain condition right nonsingular semiperfect right CS-rings are precisely direct sums of blocked triangular matrix rings (Theorem 3.9). In particular, a complete structure of such rings is obtained when the Jacobson radical is nil (Theorem 4.3). These rings turn out to be precisely the right nonsingular semiprimary right CS-rings characterized by Chatters and Hajarnavis (Theorem 3.1 of [1]). The proof of our main results depends upon several key lemmas. Lem-

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ma 3.6 is also of independent interest. This lemma shows that for $n > 1$, the $n \times n$ matrix ring over a local right Ore domain K is a right CS-ring if and only if K is a right and left valuation domain. This extends the well known result (Lemma 12.8 and Corollary 12.10 of [2]) that the $n \times n$ matrix ring, $n > 1$, over a commutative domain K is a right CS-ring if and only if K is a Prüfer domain (note local Prüfer domains are precisely commutative valuation domains).

2. NOTATION AND PRELIMINARIES

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right unital. For any two R -modules M and N , M is said to be N -injective if for every submodule K of N , any R -homomorphism $\phi: K \rightarrow M$ can be extended to an R -homomorphism $\hat{\phi}: N \rightarrow M$. Equivalently, for any R -homomorphism $\phi: N \rightarrow E(M)$, where $E(M)$ is the injective hull of M , $\phi(N) \subset M$. M is said to be self-injective if it is M -injective. Self-injective modules have traditionally been called quasi-injective modules. A submodule K of M is said to be essential in M , denoted by $K \subset_e M$, if for any non-zero submodule L of M , $K \cap L \neq 0$. M is called a CS (or extending) module if every submodule of M is essential in a direct summand of M , equivalently, if every closed submodule of M is a direct summand of M . If M has finite uniform dimension, then M is CS if and only if every uniform closed submodule of M is a direct summand of M (Corollary 7.8 of [2]).

A ring R is semiperfect if it has a complete set $\{e_i\}_{1 \leq i \leq n}$ of primitive orthogonal idempotents such that each $e_i R e_i$ is a local ring. R is said to be right nonsingular if its right singular ideal $Z(R) = \{r \in R \mid rI = 0 \text{ for some essential right ideal } I \text{ of } R\}$ is zero. R is said to be a right CS-ring if the right module R_R is right CS. The term regular ring will mean von Neumann regular ring. R is said to be a right valuation ring if for any two right ideals I and J either $I \subset J$ or $J \subset I$. A left valuation ring is defined similarly.

As is customary $Q(R)$ will denote the right maximal quotient ring of R . It is well known that for a right nonsingular ring R with finite uniform dimension, $Q(R)$ is semisimple artinian. The lattice of closed right ideals $L^*(Q(R))$ of $Q(R)$ is isomorphic to the lattice of closed right ideals $L^*(R)$ of R under the correspondence $A \rightarrow A \cap R$. It follows that the uniform closed right ideals of R are precisely those of the form $eQ \cap R$ where eQ is any minimal right ideal of Q . Whenever $Q(R)$ is simple artinian, say $M_n(D)$, then any minimal (equivalently uniform closed), right ideal I of

$Q(R)$ is of the form

$$I = \left\{ \left(\begin{array}{cccc} a_1x_1 & a_1x_2 & \cdots & a_1x_n \\ a_2x_1 & a_2x_2 & \cdots & a_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_nx_1 & a_nx_2 & \cdots & a_nx_n \end{array} \right) \middle| x_j \in D, 1 \leq j \leq n \right\}$$

for some fixed $a_i \in D$, $1 \leq i \leq n$. This fact will be frequently used.

3. SEMIPERFECT RINGS SATISFYING (*)

Throughout this section, unless otherwise stated, R will denote a semiperfect right nonsingular ring and $Q = Q(R)$, its right maximal quotient ring. We will always denote by $\{e_i\}_{1 \leq i \leq n}$ a complete set of orthogonal idempotents in R . We begin with a simple fact.

LEMMA 3.1. *If R is a right CS-ring, then each e_iR is a uniform right ideal and e_iRe_i is a local domain.*

Proof. Since an indecomposable, CS-module is uniform, it follows that e_iR is uniform. Further, e_iRe_i is a domain because R is nonsingular. ■

Suppose R is a right CS-ring. Write $R = (\oplus_{i \in I_1} e_iR) \oplus (\oplus_{i \in I_2} e_iR) \oplus \cdots \oplus (\oplus_{i \in I_k} e_iR)$, where for all $i \in I_u$ and $j \in I_v$, $e_iQ \simeq e_jQ$ if and only if $u = v$. It is straightforward to see that $R = (\oplus_{i \in I_1} e_iR) \oplus (\oplus_{i \in I_2} e_iR) \oplus \cdots \oplus (\oplus_{i \in I_k} e_iR)$ is a ring decomposition of R . For convenience we will assume henceforth that R is indecomposable. So, Q is simple artinian, and $e_iQ \simeq e_jQ$ for all i, j .

The following lemma is well known (Theorem 7.3 of [2]). We give the proof for the sake of completeness.

LEMMA 3.2. *If R is right CS and e_iR is not embeddable in e_jR then e_iR is e_jR -injective.*

Proof. Suppose M is a submodule of e_jR and let $f: M \rightarrow e_iR$ be an R -homomorphism. Let $U = \{x - f(x): x \in M\} \subset e_iR \oplus e_jR$. Then $U \cap e_iR = 0$. Since $e_iR \oplus e_jR$ is CS, there exists a direct summand U^* of $e_iR \oplus e_jR$ such that $U \subset_e U^*$. By the Krull-Schmidt theorem, $e_iR \oplus e_jR = U^* \oplus e_iR$ or $U^* \oplus e_jR$. If $e_iR \oplus e_jR = U^* \oplus e_jR$, then, because $U \cap e_iR = 0$, we have $e_iR \simeq U^*$ is embeddable in e_jR , a contradiction. Therefore, $e_iR \oplus e_jR = U^* \oplus e_iR$. Let π be the projection map from $U^* \oplus e_iR$ onto e_iR . Then, $\pi|_{e_jR}$ is the required extension of f and the proof is complete. ■

diction again. Hence, $U \cap R$ does not contain an idempotent, a contradiction to the fact that R is a right CS-ring. ■

Suppose R is a right CS-ring satisfying the condition (*). We divide the set $\{e_i R \mid 1 \leq i \leq n\}$ of uniform right ideals into equivalence classes by defining $e_i R \sim e_j R$ if and only if $e_i R \simeq e_j R$. By renumbering, if necessary, we can assume $[e_1 R], [e_2 R], \dots, [e_k R]$ are equivalence classes where $[e_i R]$ is the direct sum of uniform right ideals $e_j R$ in the family $\{e_l R \mid 1 \leq l \leq n\}$ such that $e_i R \simeq e_j R$. Then, $R = [e_1 R] \oplus [e_2 R] \oplus \dots \oplus [e_k R]$. So, by Lemma 3.4, for $1 \leq i, j \leq k$ either $e_i R e_j = 0$ or $e_j R e_i = 0$. By renumbering again, if necessary, we can assume that for $1 \leq i < j \leq k$, $e_j R e_i = 0$. Under this ordering of the family $\{e_i R\}$, we have the following:

COROLLARY 3.5. *If R is a right CS-ring satisfying the condition (*), then, for $1 \leq i < j \leq k$, $e_i R e_j \simeq e_i Q e_j$ (as additive groups). In particular, for $1 \leq i < j \leq k$, $e_i R e_j \neq 0$.*

Proof. By Lemma 3.2, $e_i R$ is $e_j R$ -injective. So, $e_i R e_j \simeq e_i Q e_j$. ■

Recall that a ring R is called right valuation if for any two right ideals I and J of R either $I \subset J$ or $J \subset I$. It is not hard to see that a right Ore domain K with D as its right classical quotient ring is a right and left valuation domain if and only if for every $a \in D$ either $a \in K$ or $a^{-1} \in K$.

It is known that if K is a right Ore domain and if $M_n(K)$, $n > 1$, is a right CS-ring, then K is left Ore (Corollary 12.9 of [2]). The following lemma gives equivalent conditions under which the $n \times n$ matrix ring $M_n(K)$, $n > 1$, over a local right Ore domain is right CS.

LEMMA 3.6. *Suppose K is a local right Ore domain with right classical quotient ring D and $n > 1$. Then, the following are equivalent:*

- (i) $M_n(K)$ is a right CS-ring,
- (ii) K is a right and left valuation domain.

Proof. Since K is a right and left valuation domain if and only if for every $c \in D$, either $c \in K$ or $c^{-1} \in K$, it is sufficient to show that $M_n(K)$ is a right CS-ring if and only if for every $c \in D$, either $c \in K$ or $c^{-1} \in K$. Firstly, assume $M_n(K)$ is right CS. Let $0 \neq c \in D$. Consider the right ideal

$$U = \left\{ \left(\begin{array}{cccc|c} a_1 & a_2 & \cdots & \cdots & a_n \\ ca_1 & ca_2 & \cdots & \cdots & ca_n \\ 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \end{array} \right) \mid a_i \in D, 1 \leq i \leq n \right\}$$

$$= \left\{ \sum_{i=1}^n a_i e_{1i} + \sum_{i=1}^n ca_i e_{2i} \mid a_i \in D, 1 \leq i \leq n \right\}.$$

diction again. Hence, $U \cap R$ does not contain an idempotent, a contradiction to the fact that R is a right CS-ring. ■

Suppose R is a right CS-ring satisfying the condition (*). We divide the set $\{e_i R \mid 1 \leq i \leq n\}$ of uniform right ideals into equivalence classes by defining $e_i R \sim e_j R$ if and only if $e_i R \simeq e_j R$. By renumbering, if necessary, we can assume $[e_1 R], [e_2 R], \dots, [e_k R]$ are equivalence classes where $[e_i R]$ is the direct sum of uniform right ideals $e_j R$ in the family $\{e_l R \mid 1 \leq l \leq n\}$ such that $e_i R \simeq e_j R$. Then, $R = [e_1 R] \oplus [e_2 R] \oplus \dots \oplus [e_k R]$. So, by Lemma 3.4, for $1 \leq i, j \leq k$ either $e_i R e_j = 0$ or $e_j R e_i = 0$. By renumbering again, if necessary, we can assume that for $1 \leq i < j \leq k$, $e_j R e_i = 0$. Under this ordering of the family $\{e_i R\}$, we have the following:

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Proof. By Lemma 3.2, $e_i R$ is $e_j R$ -injective. So, $e_i R e_j \simeq e_i Q e_j$. ■

Recall that a ring R is called right valuation if for any two right ideals I and J of R either $I \subset J$ or $J \subset I$. It is not hard to see that a right Ore domain K with D as its right classical quotient ring is a right and left valuation domain if and only if for every $a \in D$ either $a \in K$ or $a^{-1} \in K$.

It is known that if K is a right Ore domain and if $M_n(K)$, $n > 1$, is a right CS-ring, then K is left Ore (Corollary 12.9 of [2]). The following lemma gives equivalent conditions under which the $n \times n$ matrix ring $M_n(K)$, $n > 1$, over a local right Ore domain is right CS.

LEMMA 3.6. *Suppose K is a local right Ore domain with right classical quotient ring D and $n > 1$. Then, the following are equivalent:*

- (i) $M_n(K)$ is a right CS-ring,
- (ii) K is a right and left valuation domain.

Proof. Since K is a right and left valuation domain if and only if for every $c \in D$, either $c \in K$ or $c^{-1} \in K$, it is sufficient to show that $M_n(K)$ is a right CS-ring if and only if for every $c \in D$, either $c \in K$ or $c^{-1} \in K$. Firstly, assume $M_n(K)$ is right CS. Let $0 \neq c \in D$. Consider the right ideal

$$U = \left\{ \left(\begin{array}{cccccc} a_1 & a_2 & \dots & \dots & a_n \\ ca_1 & ca_2 & \dots & \dots & ca_n \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 \end{array} \right) \mid a_i \in D, 1 \leq i \leq n \right\}$$

$$= \left\{ \sum_{i=1}^n a_i e_{1i} + \sum_{i=1}^n ca_i e_{2i} \mid a_i \in D, 1 \leq i \leq n \right\}.$$

U is a uniform closed right ideal of $M_n(D)$. So, $U \cap M_n(K)$ is a non-zero closed right ideal of $M_n(K)$. Since $M_n(K)$ is right CS, $U \cap M_n(K)$ must contain an idempotent, say, $x = \sum_{i=1}^n a_i e_{1i} + \sum_{i=1}^n c a_i e_{2i}$. Observe that at least one of a_1 and a_2 is non-zero. Since $x \in M_n(K)$, $c a_1 \in K$. Also, because x is an idempotent $a_1^2 + a_2 c a_1 = a_1$. Assume $a_1 \neq 0$. If a_1 is invertible in K , then $c \in K$. If a_1 is not invertible in K , then since K is local, $1 - a_1$ is invertible in K . Also then we have $a_1 + a_2 c = 1$ (in D). Thus, $c^{-1} = (1 - a_1)^{-1} a_2 \in K$. If $a_1 = 0$, then a_2 must be non-zero. Using once again $x = x^2$, we get $a_2 c a_2 = a_2$, so that $c a_2 = a_2 c = 1$. Thus, $c^{-1} = a_2 \in K$.

Conversely, let (ii) hold. Since $M_n(K)$ has finite uniform dimension, it is sufficient to show that every uniform closed right ideal of $M_n(K)$ contains an idempotent and hence is a summand of $M_n(K)$. Any uniform closed right ideal of $M_n(K)$ is of the form $U \cap M_n(K)$, where U is a uniform closed (indeed minimal) right ideal of $M_n(D)$. Now, any uniform closed right ideal U in $M_n(D)$ is of the form

$$\left\{ \left(\begin{array}{cccc} a_1 x_1 & a_1 x_2 & \cdots & a_1 x_n \\ a_2 x_1 & a_2 x_2 & \cdots & a_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n x_1 & a_n x_2 & \cdots & a_n x_n \end{array} \right) \middle| \text{for } 1 \leq i \leq n, x_i \in D \right\},$$

where a_i , $1 \leq i \leq n$ are fixed elements of D . Thus, any uniform closed right ideal of $M_n(K)$ has the form

$$\begin{aligned} U_1 &= \left\{ \left(\begin{array}{cccc} a_1 x_1 & a_1 x_2 & \cdots & a_1 x_n \\ a_2 x_1 & a_2 x_2 & \cdots & a_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n x_1 & a_n x_2 & \cdots & a_n x_n \end{array} \right) \middle| \text{for } 1 \leq i, j \leq n, x_j \in D, a_i x_j \in K \right\} \\ &= \left\{ \sum_{1 \leq i, j \leq n} a_i x_j e_{ij} \mid \text{for } 1 \leq i, j \leq n, x_j \in D, a_i x_j \in K \right\} \end{aligned}$$

for some fixed elements a_i , $1 \leq i \leq n$ of D .

Suppose, $0 \neq x = \sum_{1 \leq i, j \leq n} a_i x_j e_{ij} \in U_1$. Among a_1, a_2, \dots, a_n , let p be the largest integer such that $a_p \neq 0$. If for all i , $1 \leq i \leq p$, $a_i a_p^{-1} \in K$ then taking $x_p = a_p^{-1}$, $x_i = 0$ for $i \neq p$, we get an idempotent element in U_1 and so we are done. Therefore, let there exist i such that $a_i a_p^{-1} \notin K$. Let i_0 be the smallest integer such that $a_{i_0} a_p^{-1} \notin K$. Then, by hypothesis, $a_p a_{i_0}^{-1} \in K$. Since by choice of i_0 , $a_i a_p^{-1} \in K$ for $i < i_0$, it follows that $a_i a_{i_0}^{-1} \in K$ for $i \leq i_0$ and $i = p$.

Now, if $a_i a_{i_0}^{-1} \in K$ for all i , then once again we are done because we can take $x_{i_0} = a_{i_0}^{-1}$ and $x_i = 0$ for $i \neq i_0$. If there exists i such that $a_i a_{i_0}^{-1} \notin K$, then once again choose the smallest integer i_1 such that $a_{i_1} a_{i_0}^{-1} \notin K$. Notice that $i_0 < i_1 < p$. By hypothesis, $a_{i_0} a_{i_1}^{-1} \in K$. But then, as above, $a_i a_{i_1}^{-1} \in K$ for $i \leq i_1$ and $i = p$. If $a_i a_{i_1}^{-1} \in K$ for all i , then we are finished. If not, then we can proceed as above. This process will terminate after a finite number of steps and thus the proof is completed. ■

THEOREM 3.7. *Suppose R is a right CS-ring satisfying the condition (*). Then there exists a division ring D and positive integers n_1, n_2, \dots, n_k such that*

$$R \simeq \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix},$$

where for each i , $1 \leq i \leq k$, D_i is a local domain contained in D . Furthermore, if for any i , $n_i > 1$ then (i) D is right and left classical quotient ring of D_i , (ii) for every $c \in D$ either $c \in D_i$ or $c^{-1} \in D_i$, i.e., D_i is a right and left valuation domain, and (iii) $M_{n_i}(D_i)$ is a right CS-ring. In general, D_k is a right Ore domain with D as its classical right quotient ring.

Proof. Let n_i be the number of direct summands in $[e_i R]$. Let $Q = Q(R)$, the right maximal quotient ring of R . Let $D = e_k Q e_k$. Then D is a division ring and for all $i < j$, $D \simeq e_i Q e_j \simeq e_i R e_j$, as additive groups. Since for each i , $e_i R e_i$ is a subring of $e_i Q e_i$ and $e_i Q e_i \simeq e_k Q e_k$ (as rings), it follows that $e_i R e_i$ is embeddable in D . Let D_i be a copy of $e_i R e_i$ in D . Then for each i , D_i is a local domain. Thus,

$$R \simeq \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix}.$$

Now, let $n_i > 1$ for some i . We will show that for every $c \in D$ either $c \in D_i$ or $c^{-1} \in D_i$. From this it will immediately follow that D is both a right and left classical quotient ring of D_i . Let, if possible, there exist

$c \in D$ such that neither $c \in D_i$ nor $c^{-1} \in D_i$. Let $t = \sum_{j=1}^{i-1} n_j + 1$ and let

$$U = \left\{ \left(\begin{array}{cccccc} 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \\ a_{t1} & a_{t2} & \cdots & \cdots & a_{tn} \\ ca_{t1} & ca_{t2} & \cdots & \cdots & ca_{tn} \\ 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \end{array} \right) \mid a_{ij} \in D, 1 \leq j \leq n \right\}$$

$$= \left\{ \sum_{j=1}^n a_{tj} e_{tj} + \sum_{j=1}^n ca_{tj} e_{t+1j} \mid a_{tj} \in D, 1 \leq j \leq n \right\}.$$

Then, U is a uniform closed right ideal of Q . Thus, $U \cap R$ is a uniform closed right ideal of R . It can be seen, by using the argument as in Lemma 3.4, that $U \cap R$ does not contain an idempotent, a contradiction to the fact that R is a right CS-ring. The fact that $M_{n_i}(D_i)$ is a right CS-ring follows from Lemma 3.6.

We, now, show that if $n_k = 1$, then D_k is a right Ore domain with D as its classical right quotient ring. We claim that $Q(D_k)$ is a division ring. In general, $Q(D_k)$ is a simple regular self-injective ring. If $Q(D_k)$ is not a division ring, then $Q(D_k)$ must contain an infinite family of orthogonal idempotents. This infinite family of orthogonal idempotents induces an infinite family of orthogonal idempotents in the simple artinian ring

$$Q = Q(R) = \begin{pmatrix} * & * \\ * & Q(D_k) \end{pmatrix}$$

(Proposition 3.3 of [5]), a contradiction. Thus, $Q(D_k)$ is a division ring, and so D_k is a right Ore domain. But, then, $Q(D_k)$ is the right classical ring of quotients of D_k and is embeddable in D .

We will now show that $\dim_{Q(D_k)} D = 1$. If possible, let $\dim_{Q(D_k)} D = r > 1$. Let

$$R_1 = \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) \end{pmatrix},$$

$$V = \begin{pmatrix} M_{n_1 \times n_k}(D) \\ M_{n_2 \times n_k}(D) \\ \cdots \\ M_{n_{k-1} \times n_k}(D) \end{pmatrix}.$$

Then,

$$R = \begin{pmatrix} R_1 & V \\ 0 & D_k \end{pmatrix}.$$

Define $\phi: \begin{pmatrix} R_1 & V \\ 0 & D_k \end{pmatrix} \rightarrow M_s(Q(D_k))$, where $s = (\sum_{j=1}^{k-1} n_j)r + 1$, by

$$\phi \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 & \cdots & 0 & v_1 \\ 0 & a & \cdots & 0 & v_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & v_r \\ 0 & 0 & \cdots & 0 & b \end{pmatrix}.$$

Then, ϕ is a ring monomorphism and $R \simeq \text{Im } \phi \subset {}_e M_s(Q(D_k))$. It follows that $Q(R) \simeq M_s(Q(D_k))$. Thus, $u \cdot \dim(Q(R)) = s > u \cdot \dim(R)$, a contradiction. Hence, $\dim_{Q(D_k)} D = 1$. Thus, $Q(D_k) = D$. ■

THEOREM 3.8. *Suppose D is a division ring and n_1, n_2, \dots, n_k are positive integers. Let*

$$R = \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_k-1}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix},$$

where for each i , $1 \leq i \leq k$, D_i is a local domain contained in $D = Q(D_k)$. Assume that if $n_i > 1$ then D_i is a right Ore domain with right classical quotient ring D and for every $c \in D$ either $c \in D_i$ or $c^{-1} \in D_i$. Then R is a semiperfect right nonsingular right CS-ring satisfying the condition (*).

Proof. Clearly, R is semiperfect right nonsingular ring. To prove R satisfies the condition (*), we write $R = \oplus \sum_{1 \leq i \leq n} e_{ii} R$, where $\{e_{ij}\}$ are matrix units. Combining the isomorphic rows we can write $R = [e_1 R] \oplus [e_2 R] \oplus \cdots \oplus [e_n R]$, where $[e_i R]$ contains the rows in R in which the i th diagonal block lies. Observe that for $i < j$, $e_j R e_i = 0$ and $e_i R e_j \neq 0$. So, the condition (*) is satisfied vacuously.

We will show that R is a right CS-ring. Since R has finite uniform dimension, it is sufficient to prove that every uniform closed right ideal of R contains an idempotent. Since $M_n(D)$, $n = \sum_{j=1}^k n_j$, is a regular self-injective ring and $R \subset {}_e M_n(D)$, we have $Q = Q(R) = M_n(D)$.

We will now show that any uniform closed right ideal of Q contains an idempotent of R . So, let

$$U = \left\{ \left(\begin{array}{cccc} a_1 x_1 & a_1 x_2 & \cdots & a_1 x_n \\ a_2 x_1 & a_2 x_2 & \cdots & a_2 x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n x_1 & a_n x_2 & \cdots & a_n x_n \end{array} \right) \mid \text{for } 1 \leq i \leq n, x_i \in D \right\},$$

where $a_i \in D$, $1 \leq i \leq n$ are fixed, be a uniform closed right ideal of Q .

Among a_1, a_2, \dots, a_n , let p be the largest integer such that $a_p \neq 0$. Let $p = \sum_{j=1}^{i-1} n_j + r$, where $1 \leq r \leq n_i$.

Case 1. $p = \sum_{j=1}^{i-1} n_j + 1$.

In this case taking $x_p = a_p^{-1}$, $x_i = 0$ for $i \neq p$, we get an idempotent in $U \cap R$.

Case 2. $p = \sum_{j=1}^{i-1} n_j + r$ where $1 < r \leq n_i$.

In this case, as in Lemma 3.6, there exist t such that $\sum_{j=1}^{i-1} n_j + 1 < t \leq \sum_{j=1}^{i-1} n_j + r = p$ and $a_k a_t^{-1} \in D_i$ for all k , $\sum_{j=1}^{i-1} n_j + 1 \leq k \leq p$. But then by taking $x_t = a_t^{-1}$, $x_i = 0$ for $i \neq t$, we get an idempotent element of R , completing the proof. ■

Combining Theorem 3.7 and Theorem 3.8, we have

THEOREM 3.9. *Suppose R is an indecomposable right-nonsingular semiperfect ring satisfying the condition (*). Then R is a right CS-ring if and only if there exists a division ring D and positive integers n_1, n_2, \dots, n_k such that*

$$R \simeq \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix},$$

where for each i , D_i is a local domain contained in D , $Q(D_k) = D$. Furthermore, if for any i , $n_i > 1$, then (i) D_i is the right and left Ore domain with classical quotient ring D , (ii) for each $c \in D$, either $c \in D_i$ or $c^{-1} \in D_i$, and (iii) $M_{n_i}(D_i)$ is a right CS-ring.

We now give an example to show that if R does not satisfy the condition (*), then R need not have the blocked triangular structure as described above.

EXAMPLE 3.1. Suppose p is a prime and $\mathbb{Z}_{(p)}$, the localization of \mathbb{Z} at p . Let

$$\begin{aligned} R &= \begin{pmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} \end{pmatrix} \\ &= e_1R \oplus e_2R \end{aligned}$$

where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, R is an indecomposable semiperfect right nonsingular ring. Also,

$$Q(R) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$$

so that $e_1Q(R)e_2 = \mathbb{Q} = e_2Q(R)e_1$. Since for any $c \in \mathbb{Q} = Q(\mathbb{Z}_{(p)})$, either $c \in \mathbb{Z}_{(p)} = e_2Re_1$ or $c^{-1} \in p\mathbb{Z}_{(p)} = e_1Re_2$. Thus, R does not satisfy the condition (*). It can be easily seen that R is a right CS-ring.

4. SEMIPERFECT RINGS WITH NIL RADICAL

In this section we will apply Theorem 3.9 to semiperfect right nonsingular right CS-rings with nil Jacobson radical. In this case the condition (*) is vacuously satisfied (Lemma 4.2). The set $\{e_i\}_{1 \leq i \leq n}$, as before, will denote a complete set of primitive orthogonal idempotents. We begin with the following lemma.

LEMMA 4.1. *For a right nonsingular semiperfect right CS-ring R with nil Jacobson radical, the e_iR are uniform and e_iRe_i are division rings.*

Proof. As stated in Lemma 3.1, the e_iR are uniform and e_iRe_i are local domains. But, since e_iJe_i is nil, e_iRe_i is a division ring. ■

LEMMA 4.2. *Let R be a right nonsingular semiperfect right CS-ring with nil Jacobson radical. Then, $e_iR \cong e_jR$ implies $e_iRe_j = 0$ or $e_jRe_i = 0$.*

Proof. Let $\phi: e_iR \rightarrow e_jR$ and $\psi: e_jR \rightarrow e_iR$ be non-zero homomorphisms. Then, $0 \neq \psi\phi \in e_iRe_i$. Since, e_iRe_i is a division ring, $\psi\phi$ is onto. This implies ψ is onto. Since R is right nonsingular and e_jR is uniform, ψ is also one-one. Thus, $e_iR \cong e_jR$, a contradiction. ■

As in the previous section, we write $R = [e_1R] \oplus [e_2R] \oplus \cdots \oplus [e_kR]$, by dividing the family $\{e_1R, e_2R, \dots, e_nR\}$ of uniform right ideals into equivalence classes. In view of Lemma 4.2, we can assume, by renumbering, if necessary, that for $1 \leq i < j \leq k$, $e_jRe_i = 0$.

The following theorem gives the structure of semiperfect right nonsingular right CS-rings with nil Jacobson radical.

THEOREM 4.3. *Let R be an indecomposable right nonsingular semiperfect CS-ring. Then the following are equivalent:*

- (a) *The Jacobson radical of R is nil.*
- (b) *R is semiprimary.*
- (c) *There exists a division ring D , and positive integers n_1, n_2, \dots, n_k such that*

$$R \cong \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix},$$

where for each i , $1 \leq i \leq k$, D_i is a division subring of D . Furthermore, if for some i , $n_i > 1$ or $i = k$ then $D_i = D$.

Proof. We first prove (a) \Rightarrow (c). So, assume that the Jacobson radical of R is nil. With the above renumbering of e_i 's, we have $e_j R e_i = 0$, $1 \leq i < j \leq k$. Therefore, R satisfies the condition (*) vacuously. So, by Theorem 3.9,

$$R \cong \begin{pmatrix} M_{n_1}(D_1) & M_{n_1 \times n_2}(D) & \cdots & M_{n_1 \times n_{k-1}}(D) & M_{n_1 \times n_k}(D) \\ 0 & M_{n_2}(D_2) & \cdots & M_{n_2 \times n_{k-1}}(D) & M_{n_2 \times n_k}(D) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_{k-1}}(D_{k-1}) & M_{n_{k-1} \times n_k}(D) \\ 0 & 0 & \cdots & 0 & M_{n_k}(D_k) \end{pmatrix},$$

where $D = e_k Q(R) e_k = Q(D_k)$ is a division ring, n_1, n_2, \dots, n_k are positive integers, $D_i = e_i R e_i$ is a local domain contained in D . However, D_i is, indeed, a division ring by Lemma 4.1. Now, if for some i , $n_i > 1$ then, by Lemma 3.7, for every $c \in D$ either $c \in D_i$ or $c^{-1} \in D_i$. So, $D_i = D$. Furthermore, $D = Q(D_k) = D_k$. This proves (c). The implication (c) \Rightarrow (b) follows by [1, Theorem 3.1] and the implication (b) \Rightarrow (a) is obvious. ■

The following corollary is now an immediate consequence.

COROLLARY 4.4. *Suppose R is an indecomposable right nonsingular semiperfect right CS-ring with nil Jacobson radical. Then, with the notations of*

Theorem 4.3, R is (i) right artinian if and only if for $i > 1$, D is finite dimensional as a right vector space over D_i ; (ii) left artinian if and only if for every i , D is finite dimensional as a left vector space over D_i ; (iii) right serial if and only if for $i > 1$, $D_i = D$; and (iv) left serial if and only if for all i , $D_i = D$.

Remark 4.1. In Example 3.1, the Jacobson radical is not nil. Thus, if the Jacobson radical is not nil, then an indecomposable right nonsingular semiperfect right CS-ring need not have a blocked triangular structure as obtained in Theorem 4.3.

REFERENCES

1. A. W. Chatters and C. R. Hajarnavis, Rings in which every complement right ideal is a direct summand, *Quart. J. Math. Oxford* **28** (1977), 61–80.
2. Nguyen Viet Dung, Dinh Van Huynh, P. F. Smith, and R. Wisbauer, “Extending Modules,” Pitman, London, 1994.
3. Manabu Harada, “Factor Categories with Applications to Direct Decomposition of Modules,” Lecture Notes in Pure and Appl. Math., Vol. 88, Dekker, New York, 1983.
4. B. L. Osofsky and P. F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, *J. Algebra* **139** (1991), 342–354.
5. Kazunori Sakano, Maximal quotients rings of generalized matrix rings, *Comm. Algebra* **12** (1984), 2055–2065.