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RING OF MORITA CONTEXT IN WHICH EACH RIGHT IDEAL IS WEAKLY SELF-INJECTIVE

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ABSTRACT. In this paper, among others, an example of a noetherian ring of Morita Context in which each right ideal is weakly self-injective, has been studied.

1. INTRODUCTION

A right R-module M is said to be self-injective if for every R-homomorphism $\varphi : M \to E(M)$, from M into its injective hull $E(M)$, $\varphi(M) \subseteq M$. Similarly, M is weakly self-injective if for every homomorphism $\varphi : M \to E(M)$, there exists $X \subseteq M$ such that $\varphi(M) \subseteq X \cong M$. If K is a right Ore domain then each right ideal is clearly weakly self-injective. Let K be a right and left noetherian domain. We consider the ring

$$R = \begin{pmatrix} K & K^* & 0 & \ldots & 0 & 0 \\ 0 & K & K^* & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & K & K^* \\ K^* & 0 & 0 & \ldots & 0 & K \end{pmatrix}$$

of all $n \times n$ matrices whose only nonzero possible entries are on the main diagonal and at $(1,2), (2,3), \ldots, (n-1,n), (n,1)$ places. The entries on the main diagonal belong to K and on the $(i,i+1)$th places, for all $i, 1 \leq i \leq n$, belong to a null K-algebra K^* of rank 1. We prove that each right and left ideal in R is weakly self-injective and address some related questions by considering K-algebra K^* of rank greater than 1. Rings whose right ideals are self-injective have been studied in great detail by many authors. It is interesting to note that the injective hull of R is a ring with the property that each right ideal is self-injective([1]).

Throughout this paper, by a module we mean a right module. K will denote a right noetherian domain. Q, the right maximal ring of quotients of K. K^*, a null algebra of rank 1 over K that is, $KK^* = 0 = K^*K$. We shall identify K^* with K as additive abelian groups.

For any module $M, E(M)$ will denote the injective hull of M. For any two sets I and J, $(IJ)_i$ will denote the set of $n \times n$ matrices with (i,i)th element from I, $(i,i+1)$th element from J and all other elements zero. Note that in our notation , $R = \sum_{i=1}^{n}(KK^*)_i$. The fact that R is a ring under usual matrix operations is clear, because $KK^* = 0 = K^*K$.

2. Main Result

We begin with a lemma whose proof is straightforward.

Lemma 2.1. *If K is noetherian then R is noetherian.*

Proof. Writing R as

$$R = \begin{pmatrix} T & P \\ Q & K \end{pmatrix}$$

where

$$T = \begin{pmatrix} K & K^* & 0 & \dots & 0 & 0 \\ 0 & K & K^* & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & K^* \\ 0 & 0 & 0 & \dots & \dots & K \end{pmatrix},$$

$$P = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ \dots \\ K^* \end{pmatrix}, Q = \begin{pmatrix} K^* & 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

Then, A is a two-sided ideal of R and $\frac{R}{A} = T \times K$. Since both T and K are noetherian as $\frac{R}{A}$- modules, $(R/A)_{R/A}$ is noetherian. So R/A is noetherian as R-module. Hence, it is enough to show A_R is noetherian. Let $P^* = \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}$ and $Q^* = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$. Then both Q and P are two-sided ideals of A, infact of R. Also note that, φ: $Q^* \rightarrow K^*$ defined by $\varphi\begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ k^* & \dots & 0 \end{pmatrix} = k^*$ is a K-module isomorphism.

Hence Q^*_K is noetherian. But K is embeddable in R via the map $k \to \begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & k \end{pmatrix}$. So Q^*_R is noetherian. Hence $(A/P^*)_R = Q^*_R$ is noetherian. Also P^*_R is noetherian. Hence, A_R is noetherian, which proves the result. ∎

Lemma 2.2. *The right ideals of the ring R are precisely $\sum_{k=1}^{n}(I_k J_k)_k$, where*

$$\sum_{k=1}^{n}(I_k J_k)_k = \begin{pmatrix} I_1 & J_1 & 0 & \dots & 0 \\ 0 & I_2 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ J_n & 0 & 0 & \dots & I_n \end{pmatrix}$$

and for each k, $1 \le k \le n$, I_k and J_k are right ideals of K such that $I_k \subseteq J_k$.

Proof. Clearly, $\sum_{k=1}^{n}(I_k J_k)_k$ is a right ideal of R whenever I_k and J_k are right ideals of K such that $I_k \subseteq J_k$. Conversely, if I is a right ideal of R, then I is finitely generated. We shall prove that I is of the form $\sum_{k=1}^{n}(I_k J_k)_k$,by induction on the number m of generators of I. For $m=1$, suppose that I is generated by $a = \sum_{k=1}^{n}\left(a_{ii}e_{ii} + a_{ii+1}e^*_{ii+1}\right)$. Then,

$$\begin{aligned} I &= aR \\ &= \sum_{i=1}^{n}\left(a_{ii}a_{ii+1} + a_{ii+1}e^*_{ii+1}\right)R \\ &= \begin{pmatrix} a_{11}K & a_{11}K^* + a^*_{12}K & 0 & 0 & \dots & 0 \\ 0 & a_{22}K & a_{22}K^* + a^*_{23}K & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a^*_{n1}K + a_{nn}K^* & 0 & \dots & 0 & \dots & a_{nn}K \end{pmatrix} \end{aligned}$$

which is clearly of the form $\sum_{k=1}^{n}(I_k J_k)_k$ with $I_k \subseteq J_k$ for all $1 \le k \le n$.

Suppose the result holds for $m-1$ generators. Let I be generated by m generators $a_1, a_2, \dots, a_m$. Then $I = a_1R + a_2R + \dots + a_{m_k-1}R + a_mR$. By induction hypothesis, $a_1R + a_2R + \dots + a_{m-1}R = \sum_{k=1}^{n}(I_k J_k)_k$ and $a_mR = \sum_{k=1}^{n}(I'_k J'_k)_k$with $I_k \subseteq J_k$ and $I'_k \subseteq J'_k$ for all $1 \le k \le n$. So $I = \sum_{k=1}^{n}(I_k J_k)_k + \sum_{k=1}^{n}(I'_k J'_k)_k = \sum_{k=1}^{n}((I_k + I'_k)(J_k + J'_k))_k$ .Since

$I_k \subseteq J_k$ and $I'_k \subseteq J'_k, I_k + I'_k \subseteq J_k + J'_k$. So I has the desired form. Hence the proof is complete by induction. ∎

Lemma 2.3. *If I, J are right ideals of K with $I \subset J$ then for $1 \leq i, j \leq n$, $j \neq i, j \neq i-1 (\bmod n)$, $Hom_R((IJ)_i, (QQ)_j) = 0$, where Q is the right maximal quotient ring of K.*

Proof. Let $\phi \in Hom_R((IJ)_i, (QQ)_j)$ and $\phi(ae_{ii} + be^*_{ii+1}) = pe_{jj} + qe^*_{jj+1}$. Then $(ae_{ii}+be^*_{ii+1})e_{j+1j+1} = 0$ and $(ae_{ii}+be^*_{ii+1})e^*_{jj+1} = 0$, yields $p = 0$ and $q = 0$ respectively. ∎

Next we compute the injective hull of R_R.

Proposition 2.4. *The injective hull of R_R is*

$$\sum_{i=1}^{n}(QQ^*)_i = \begin{pmatrix} Q & Q^* & 0 & \dots & 0 & 0 \\ 0 & Q & Q^* & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & Q & Q^* \\ Q^* & 0 & 0 & \dots & 0 & Q \end{pmatrix}.$$

Proof. Suppose $0 \neq x = \sum_{i=1}^{n}(p_{ii}e_{ii} + q_{ii+1}e^*_{ii+1}) \in \sum_{i=1}^{n}(QQ)_i$. To be definite, let $p_{11} \neq 0$. For each i, $1 \leq i \leq n$, there exists $r_i \in K$ such that $p_{ii}r_i \in K, q_{ii+1}r_i \in K$ and $p_{11}r_1 \neq 0$. But then $0 \neq x\sum_{i=1}^{n} r_ie_{ii} = \sum_{i=1}^{n}(p_{ii}r_ie_{ii} + q_{ii+1}r_{i+1}e^*_{ii+1})$ is in R. Hence $\sum_{i=1}^{n}(QQ^*)_i$ is an essential extension of R.

To prove injectivity of $\sum_{i=1}^{n}(QQ^*)_i$, it is sufficient to prove that for all i, $(QQ^*)_i$ is injective as an R-module. To prove this, let $\phi : \sum_{i=1}^{n}(IJ)_k \to (QQ^*)_i$ then $\phi = \sum_{i=1}^{n}\phi_k$, where $\phi_k : (IJ)_k \to (QQ^*)_i$. By Lemma 2.3, $\phi_k = 0$ for $k \neq i, i+1$, so that $\phi = \phi_i + \phi_{i+1}$.

Define $\phi'_i : J_i \to Q$ as follows : if $b \in J_i$ and $\phi_i(ae_{ii} + be^*_{ii+1}) = pe_{ii} + qe^*_{ii+1}$ for some $a \in I_i$, then set $\phi'_i(b) = q$. The map ϕ'_i is well defined, for if, $ae_{ii} \in (IJ)_i$ and $\phi(ae_{ii}) = pe_{ii} + qe^*_{ii+1}$ then, as $ae_{ii}e_{i+1i+1} = 0$, we have $q = 0$. ϕ'_i is clearly a K-homomorphism. Since Q is injective as K-module, there exist $\alpha \in Q$ such that $\phi'_i(b) = \alpha b$ for all $b \in J_i$. If $\phi_i(ae_{ii} + be^*_{ii+1}) = pe_{ii} + qe^*_{ii+1}$ then $pe^*_{ii+1} = (pe_{ii} + qe^*_{ii+1})e_{ii+1} = \phi_i(ae^*_{ii+1})$, so that $\phi'_i(a) = p$. Thus $p = \alpha a$, $q = \alpha b$. It follows that $\phi_i(ae_{ii} + be^*_{ii+1}) = \alpha ae_{ii} + \alpha be_{ii+1} = \alpha e_{ii}(ae_{ii} + be^*_{ii+1})$. Observe, now, that if $\phi_{i+1}(ae_{i+1i+1} + be^*_{i+1i+2}) = pe_{ii} + qe^*_{ii+1}$ then, as $(ae_{i+1i+1} + be^*_{i+1i+2})e^*_{ii+1} = 0$, we have $p = 0$. If $I_{i+1} = 0$, then since $(ae_{i+1i+1} + be^*_{i+1i+2})e_{i+1i+1} = 0$ we have $\phi_{i+1} = 0$. Thus, in this case, $\phi = \phi_i$ and consequently

$$\begin{aligned}\phi(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})) &= \phi_i(ae_{ii} + be^*_{ii+1}) \\ &= \alpha\phi_i(ae_{ii} + be^*_{ii+1}) \\ &= \alpha e_{ii}(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})).\end{aligned}$$

Assume, now, that $I_{i+1} \neq 0$. Define $\phi'_{i+1} : I_{i+1} \to Q$ as follows: if $\phi_{i+1}(ae_{i+1i+1} + be^*_{i+1i+2}) = qe^*_{ii+1}$, set $\phi'_{i+1}(a) = q$. The map ϕ'_{i+1} is well defined, for if $\phi_{i+1}(be^*_{i+1i+2}) = qe^*_{ii+1}$, then, as $be^*_{i+1i+2}e_{i+1i+1} = 0$, we have $q = 0$. Clearly ϕ'_{i+1} is a K-homomorphism. As above, there exists $\beta \in Q$ such that $\phi'_{i+1}(a) = \beta a$ for all $a \in I_{i+1}$, that is, if $\phi_{i+1}(ae_{i+1i+1} + be^*_{i+1i+2}) = qe^*_{ii+1}$ then $q = \phi'_{i+1}(a) = \beta a$. Thus,

$$\phi_{i+1}(ae_{i+1i+1} + be^*_{i+1i+2}) = \beta ae^*_{ii+1} = \beta e^*_{ii+1}(ae_{i+1i+1} + be^*_{i+1i+2}).$$

Hence,

$$\begin{aligned}\phi(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})) &= \alpha e_{ii}(a_i e_{ii} + b_i e^*_{ii+1}) + \beta e^*_{ii+1}(a_{i+1}e_{i+1i+1} + b_{i+1}e^*_{i+1i+2}) \\ &= \alpha e_{ii}(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})) + \beta e^*_{ii+1}(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})) \\ &= (\alpha e_{ii} + \beta e^*_{ii+1})(\sum_{k=1}^{n}(a_k e_{kk} + b_k e^*_{kk+1})).\end{aligned}$$

Thus, in any case, there exist $\lambda \in (QQ^*)_i$ such that $\phi(x) = \lambda x$ for all $x \in I$. Consequently, for all $i, 1 \leq i \leq n$, $(QQ^*)_i$ is injective as R-module and the proof of the theorem is complete. ∎

Remark 2.1. *Müller [4] and Sakano [5] obtained the injective hull of generalized matrix. One may obtain the above result by using their method. Our construction is direct and less technical. A slight modification of the above argument yields the following theorem:*

Theorem 2.5. *If I and J are right ideals of K such that $I \subset J$ and J is non-zero then for $1 \leq i \leq n$, the injective hull of the right ideal $(IJ)_i$ is $(QQ^*)_i$.*

It is known that a domain is right weakly self-injective if and only if it is a right Ore domain (Example 1.11(ii), [3]).For the following discussion we assume that K is right Ore. Then for $n > 1$,we have the following:

Theorem 2.6. *Let K be a right Ore domain and $n > 1$. Then the ring $R = \sum_{i=1}^{n}(KK^*)_i$ is right weakly self-injective if and only if K is left Ore.*

Proof. First assume R is right weakly self-injective. Let a and b be any two non-zero elements of K, and let $x = a^{-1}e_{11} + b^{-1}e_{12}^*$. Then x is a non-zero element of $E(R_R)$. Since R is weakly self-injective, by (Remark 1.5, [3]) there exist $0 \neq y = \sum_{i=1}^{n}(a_ie_{ii}+b_ie_{ii+1}^*)$ such that $x \in yR$ and $r.ann_R(y) = 0$. If $a_i = 0$ for some i, say $i = k$, then $ye_{kk+1}^* = 0$, a contradiction to the fact that $r.ann_R(y) = 0$. Thus $a_i \neq 0$ for all i. But then y is invertible and $y^{-1} = \sum_{i=1}^{n}(a_i^{-1}e_{ii} - a_i^{-1}b_ia_{i+1}^{-1}e_{ii+1})$. Since $x \in yR$, we get $a_1^{-1}a^{-1} = \alpha$ and $a_1^{-1}b^{-1} = \beta$ for some $\alpha, \beta \in K$. Thus $\alpha a = \beta b$ for some $\alpha, \beta \in K$ and hence K is left Ore.

Conversely, assume that K is left Ore and let $0 \neq x = \sum_{i=1}^{n}(a_ie_{ii} + b_ie_{ii+1}^*)$ be an arbitrary element of $E(R_R) = \sum_{i=1}^{n}(QQ^*)_i$. Since ${}_KK$ is essential in Q, there exist $0 \neq r \in K$ such that for $1 \leq i \leq n$, ra_i, $rb_i \in K$. Let $ra_i = s_i$, $rb_i = t_i$ and $y = \sum_{i=1}^{n}r^{-1}e_{ii}$. Then y is a non-zero element of $E(R_R)$ and s_i, $t_i \in K$ for $1 \leq i \leq n$. Also $x = \sum_{i=1}^{n}(r^{-1}s_ie_{ii}+r^{-1}t_ie_{ii+1}^*) = \sum_{i=1}^{n}r^{-1}e_{ii}\sum_{i=1}^{n}(s_ie_{ii}+t_ie_{ii+1}^*) \in yR$. Moreover, if $\sum_{i=1}^{n}(x_ie_{ii} + y_ie_{ii+1}^*) \in r.ann_R(y)$ then $r^{-1}x_i = 0 = r^{-1}y_i$ for all i, $1 \leq i \leq n$. It follows that $x_i = 0 = y_i$ for all i. Consequently, $r.ann_R(y) = 0$ and therefore R is weakly self-injective. ∎

In view of the above theorem, it is clear that if K is a right noetherian domain then $R = \sum_{i=1}^{n}(KK^*)_i$ is a weakly self injective ring.

Proposition 2.7. *If K is a noetherian domain and*

$$R = \sum_{i=1}^{n}(KK^*)_i = \begin{pmatrix} K & K^* & 0 & \dots & 0 & 0 \\ 0 & K & K^* & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & K & K^* \\ K^* & 0 & 0 & \dots & 0 & K \end{pmatrix}.$$

Then every right ideal as well left of R is weakly self-injective.

Proof. Suppose $0 \neq I = \sum_{k=1}^{n}(I_kJ_k)_k$ be a right ideal of R and let Λ be the set of k, $1 \leq k \leq n$, for which I_k and J_k are not both zero. Then $I = \bigoplus\sum_{k\in\Lambda}(I_kJ_k)$ so that $E(I) = \bigoplus\sum_{k\in\Lambda}(QQ^*)_k$. Let $0 \neq \phi : I \to E(I)$ be any R-homomorphism and let for $k \in \Lambda$, $\phi_k = \pi_k\phi$, where π_k is the projection of $\bigoplus\sum_{k\in\Lambda}(QQ^*)_k$ onto $(QQ^*)_k$. Then $\phi = \sum_{k\in\Lambda}\phi_k$. Proceeding, now, as in Theorem 2.4, for all $k \in \Lambda$, we can find $\alpha_k, \beta_k \in Q$ such that $\phi_k(x) = (\alpha_ke_{kk}+\beta_{k+1}e_{kk+1}^*)x$ for all $x \in I$. Observe that, if for some $k \in \Lambda$, $k-1 \notin \Lambda$, then $\beta_k = 0$. Consequently, for all $x \in I$, $\phi(x) = \left[\sum_{k\in\Lambda}(\alpha_k e_{kk} + \beta_{k+1}e^*_{kk+1})\right]x$. Since $\phi \neq 0$, at least one of α_k's and β_k's is non-zero. Since Q is left maximal ring of quotients of K, there exist $0 \neq r \in K$ such that for all $k \in \Lambda$, $\alpha_k, \beta_k \in K$. Let $r\alpha_k = x_k$, $r\beta_k = y_k$. Now, if for some $k \in \Lambda$, $I_k \neq 0$, pick $0 \neq \lambda_k \in I_k$, and if $I_k = 0$, pick $0 \neq \lambda_k \in J_k$. Then, for all $k \in \Lambda$, $\lambda_k x_k \in I_k$, and $\lambda_k y_{k+1} \in J_k$. Consequently, for all $x \in I$,

$$\begin{aligned}\phi(x) &= (\sum_{k\in\Lambda} r^{-1}\lambda_k^{-1}e_{kk})\left(\sum_{k\in\Lambda}(\lambda_k x_k e_{kk} + \lambda_k y_{k+1}e^*_{kk+1})x\right)\\ &\in (\sum_{k\in\Lambda} r^{-1}\lambda_k^{-1}e_{kk})I\end{aligned}$$

Thus $\phi(I) \subset yI$, where $y = \sum_{k\in\Lambda} r^{-1}\lambda_k^{-1}e_{kk}$. It is easy to see that $yI \simeq I$. It follows that I is weakly self-injective. The proof is similar for left ideals of R. ∎

A slight modification of the above argument yields our next result. Before we can state it we need to mention the following definition:

Definition 2.1. *A module M is said to be weakly $N-$ injective if for every homomorphism $\varphi : N \to E(M)$, there exist $X \subseteq E(M)$ such that $\varphi(N) \subseteq X \cong M$ and weakly-injective if it is weakly N-injective for every finitely generated module N.*

Theorem 2.8. *For every n, every right ideal I of R is weakly I^n-self-injective.*

Corollary 2.9. *R is weakly-injective.*

We now give an example to show that if the null algebra K^* is of rank more than 1 then the result may not be true.

Example 2.1. *Let K be a right Ore domain. Then, neither of the rings*

$$R_1 = \begin{pmatrix} K & K^*\times K^* \\ K^*\times K^* & K \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} K & K^*\times K^* \\ K^* & K \end{pmatrix}$$

is weakly self-injective.

Proof. We prove the result for R_2. The argument for R_1 is similar. It is easy to see that $R_2 \simeq R_2'$, where

$$R_2' = \left\{ \begin{pmatrix} a & 0 & b^* \\ 0 & a & c^* \\ d^* & 0 & e \end{pmatrix} : a, b, c, d, e \in K \right\}$$

and that

$$E(R_2) \simeq \begin{pmatrix} Q & Q & Q^* \\ Q & Q & Q^* \\ Q^* & 0 & Q \end{pmatrix}$$

where Q is the right maximal quotient ring of K.

Suppose R_2 is weakly self-injective. Then for $e_{12} \in E(R_2)$, there exist $y = \begin{pmatrix} y_{11} & y_{12} & y_{13}^* \\ y_{21} & y_{22} & y_{23}^* \\ y_{31}^* & 0 & y_{33} \end{pmatrix} \in E(R_2)$ such that $e_{12} \in yR_2$ and $r.ann_{R_2}(y) = 0$. Since $e_{12} \in yR_2$, therefore,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & y_{13}^* \\ y_{21} & y_{22} & y_{23}^* \\ y_{31}^* & 0 & y_{33} \end{pmatrix} \begin{pmatrix} a & 0 & b^* \\ 0 & a & c^* \\ d^* & 0 & e \end{pmatrix}$$

for some $a, b, c, d, e \in K$. It follows that $a \neq 0$, $y_{11} = 0 = y_{21} = y_{22}$. But then $ye_{13}^* = 0$, contradicting the fact that $r.ann_{R_2}(y) = 0$. Hence R_2 is not weakly self-injective. ∎

Remark 2.2. *A similar argument can be used to show that the ring* $R' = \begin{pmatrix} D & D^2 \\ 0 & D \end{pmatrix}$, *where D is a division ring, is not weakly self-injective. It follows that every right ideal in a non-singular, artinian ring need not be weakly self-injective. However, for the ring of upper triangular matrices over a division ring , it is not hard to see that every right ideal is weakly self-injective.*

References

[1] G. Ivanov, *Non-local rings whose ideals are all quasi-injective*, Bull. Austral. Math. Soc. **6** (1972), 45-52.

[2] G. Ivanov, *Non-local rings whose ideals are all quasi-injective: addendum*, Bull. Austral. Math. Soc. **12** (1975), 159-160.

[3] S. K. Jain and S. R. López-Permouth, *A survey on the theory of weakly-injective modules*, Computational Algebra, Marcel Dekker, NY (1994), 205-232.

[4] Marianne Müller, *Rings of quotients of generalized matrix rings*, Comm. Alg. **15**(10), 1987, 1991-2015.

[5] Kazunori Sakano, *Maximal quotient rings of generalized matrix rings*, Comm. Alg. **12**(16), 1984, 2055-2065.

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