On Weakly Injective Continuous Modules

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Abstract. We study several properties of weakly injective modules. In particular, we show that, while every finitely generated weakly injective continuous module is injective, this implication does not necessarily hold for infinitely generated modules. On the other hand, weakly injective quasi-continuous modules do behave better than arbitrary weakly injective modules in the sense that any direct summand of a weakly injective quasi-continuous module is again weakly injective.

The question of when the domain of weak injectivity of certain modules is closed under direct sums and under essential extensions is also considered. This study yields characterizations of quasi-injective modules and of right SI-rings in terms of weak injectivity.

1. INTRODUCTION

The concept of weak relative injectivity of modules was introduced originally in [JL1] to study semiperfect rings for which every cyclic module is isomorphic to an essential submodule of a projective module. Since then, the study of this concept has been furthered extensively, a convenient reference for background material on the subject is [JL2]. Among other applications, weakly injective modules have served to study semiprime Goldie rings [LRY], [Z], QI rings [JLS], [LR], direct sums of matrix rings over local QF rings [JLOS] and other rings [L].

Weakly injective modules are closed under finite direct sums but not necessarily under direct summands. Similarily, essential extensions of weakly injective modules are again weakly injective but, certainly, essential submodules of weakly injective modules need not be weakly injective. Finally, while the domain of weak injectivity of a module is closed under quotient modules and submodules, it is not closed under direct sums or essential extensions. In this paper we study when the converses of the above results also hold.

It is known that a finitely generated weakly injective continuous module is indeed injective. In this paper we provide examples to show that this implication is not true in general when M is not finitely generated. Then we study rings for which the implication does hold. Rings satisfying the condition that weakly injective continuous modules are injective seem to be abundant and hard to characterize in general. We show examples of such rings which do not satisfy any obvious chain conditions as well as an example of a noetherian ring having non-injective weakly injective continuous modules.

Throughout this paper, all modules are right unital modules unless otherwise stated. For a module M, either notation $\hat{M}$ or $E(M)$ serves to denote the injective hull of M. For any other notation or concept not defined here the reader might refer to [AF] or [F].

2. WEAKLY SELF-INJECTIVE MODULES

Given two modules M and N, we say that M is weakly N-injective if for every homomorphism $\varphi : N \to E(M)$ there exists a submodule $X \subset E(M)$ which is isomorphic to M such that $\varphi(N) \subset X$. If the module M is weakly N-injective for every finitely generated module N, we say that M is weakly-injective. Additionally, in this paper we will be concerned with the following

2.1. Definition. *A module M is said to be weakly self-injective if it is weakly M-injective.*

A module M is weakly N-injective if and only if it is weakly K-injective for every submodule K of N if and only if it is weakly N/K-injective for every submodule K of N. On the other hand, the ring R of upper triangular 2×2 matrices over a field is weakly R-injective but not weakly R^2-injective. Therefore, the domain of weak injectivity of a module is not necessarily closed under (finite) direct sums. The following two propositions pertain when the domain of weak injectivity of a weakly self-injective module is closed under direct sums.

2.2. Proposition. *The following conditions are equivalent for a weakly self-injective module M:*

(1) *for any modules N and K, if M is weakly N-injective and weakly K-injective then M is weakly $N \oplus K$-injective;*
(2) *M is weakly M^n-injective for all $n \in \mathbf{Z}^+$;*
(3) *M is weakly M^2-injective.*

Proof. One needs only to show (3) $\Rightarrow$ (1). Let $\varphi : N \oplus K \to E(M)$. Considering the restrictions of φ to N and K respectively one obtains submodules X_1 and X_2

of $E(M)$, both isomorphic to M such that $\varphi(N) \subset X_1$ and $\varphi(K) \subset X_2$. From the fact that M is weakly M^2-injective one obtains that there exists a submodule $X \subset E(M)$, isomorphic to M such that $X_1 + X_2 \subset X$. It follows that $\varphi(N \oplus K) = \varphi(N) + \varphi(K) \subset X$, proving that M is indeed weakly $N \oplus K$-injective. □

2.3. Proposition. *The following conditions are equivalent for a module M:*

(1) *M is quasi-injective;*
(2) *M is weakly $M^{(\alpha)}$-injective for every cardinal α;*
(3) *M is weakly $M^{(\alpha)}$-injective for $\alpha = |End(\hat{M})|$.*

Proof. Obviously, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). It is only left to show that (3) $\Rightarrow$ (1). Consider $q(M) = End(\hat{M})(M)$, the quasi-injective hull of M. Write $q(M) = \Sigma_{\varphi \in End(\hat{M})}\ \varphi(M)$. Then $q(M) = \Phi(M^{(\alpha)})$ where $\Phi : M^{(\alpha)} \to \hat{M}$ is given by $\Phi((m_\varphi)_{\varphi \in End(\hat{M})}) = \Sigma_{\varphi \in End(\hat{M})} \varphi(m_\varphi)$. Since M is weakly $M^{(\alpha)}$-injective, there exists $X \subset \hat{M}$, isomorphic to M such that $q(M) \subset X$. But $X \subset q(M)$ therefore $q(M) = X \cong M$. Hence M is quasi-injective, as claimed. □

As a consequence of the preceeding theorem one gets that

(1) a ring R is such that every module is (weakly) N-injective whenever N is singular if and only if
(2) every singular module is semisimple.

The implicaton (2) $\Rightarrow$ (1) is trivial. For the injective case of (1) $\Rightarrow$ (2) notice that given a singular module N and a submodule T of N, since T is N-injective, $T \subset^{\oplus} N$. For the weakly injective case, notice that every singular module N is quasi-injective, by Proposition 2.3, and then proceed as in the injective case.

In the next two propositions we deal with the question of when the domains of weak injectivity of certain modules are closed under essential extensions.

2.4. Proposition. *The following conditions are equivalent for a ring R:*

(1) *for every right module M, the domain of weak injectivity of M is closed under essential extensions;*
(2) *every weakly self-injective module is injective;*
(3) *R is semisimple artinian.*

Proof. Obviously (3) $\Rightarrow$ (1). Also, if one assumes (1) then every weakly self-injective module M is weakly $E(M)$-injective and hence injective. It only rests to prove that (2) $\Rightarrow$ (3). Let R be a ring satisfying condition (2). It follows that every completely reducible module must be injective. Hence, R is a right noetherian right V-ring. Assume that R is not semisimple. Then there exists a right ideal $K \subset R$ maximal with respect to the property that $U = R/K$ is not semisimple. It follows that every quotient U/V of U with $V \neq 0$ is semisimple. Now, let X be a nonzero cyclic submodule of U. Then every homomorphism $\varphi : X \to E(X)$ either is a monomorphism or satisfies that $\varphi(X) \subset \mathrm{soc}X$. Hence X is weakly self-injective and so injective by (2). That makes U semisimple, a contradiction. Therefore, R is semisimple artinian. □

Recall that a ring R is called right SI if every singular right R-module is injective [G]. One may use the arguments above to prove the following equivalences.

2.5. Proposition. *The following conditions are equivalent for a ring R:*

(1) *for every singular right module M, the domain of weak injectivity of M is closed under essential extensions;*
(2) *every singular weakly self-injective module is injective;*
(3) *R is a right SI-ring.*

Proof. Clearly (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2). If R satisfies condition (2) then every singular semisimple module is injective. We will show that if (2) holds then every singular cyclic module is semisimple and hence injective and thus by [G] R is right SI. In order to get the argument to go through we need to start by showing that under condition (2) every cyclic singular right module is noetherian and then proceed as in the proof of the previous Proposition . Let $M = R/I$ be singular. If M is not noetherian one may consider a strictly ascending chain of submodules of M of the form

$$x_1R/I \subset (x_1R + x_2R)/I \subset \cdots .$$

Let

$$X/I = \bigcup_{k=1}^{\infty}(x_1R + \cdots + x_kR)/I.$$

Let M_1/I be a maximal submodule of x_1R/I. Since x_1R/M_1 is simple and singular, it is injective and therefore $x_1R/M_1 \subset^{\oplus} X/M_1$ and $x_1R/M_1 \subset^{\oplus} (x_1R+x_2R)/M_1 = x_1R/M_1 \oplus y_2R/M_1$, for some $y_2 \in (x_1R + x_2R)$. Take M_2/M_1 to be a maximal submodule of y_2R/M_1. Then $(x_1R+x_2R+x_3R)/M_2 \cong x_1R/M_1 \oplus y_2R/M_2 \oplus y_3R/M_2$ for some $y_3 \in x_1R + x_2R + x_3R$. Continuing this way we can produce for each $i = 2, 3, \ldots$ a submodule M_i maximal in y_iR/M_{i-1}. Let $M^* = \bigcup_{i=1}^{\infty} M_i$. Then X/M^* has a submodule N which is an infinite direct sum of singular simple modules. By (2), N is injective. But then N is a direct summand of the cyclic module M/M^*, a contradiction. Therefore M is noetherian. □

At this point, we would like to mention some related results recently obtained in [HR]:

(1) A ring R is semisimple artinian if and only if every continuous right R-module is injective.
(2) A ring R is right SI if and only if every singular continuous right R-module satisfying the restricted semisimple condition is injective.

Combining the argument in [HR] and that presented here we obtain the following conclusion:

(3) A ring R is right SI if and only if every singular continuous weakly self-injective module satisfying the restricted semisimple condition is injective.

3. WEAK INJECTIVITY AND CONTINUITY

Every weakly injective quasi-injective module is injective. Also, if a finitely generated module M is weakly injective and continuous then M is indeed injective [JL, Examples 1.11]. One could conjecture that weakly injective continuous modules in general are injective. We will show next that this is not necessarily the case even when the ring has very nice structure.

3.1. Proposition. *A right and left PCI domain R which is not a division ring has a weakly injective continuous module which is not injective.*

Proof. Let R be a right and left PCI domain which is not a division ring [C], [CF]. Since R is right and left noetherian, each submodule of $E(R_R)$ is weakly injective [LRY]. Let M be a proper submodule of $E(R_R)$ such that $E(R_R)/M$ is noetherian. Since R is not a division ring, $M \neq 0$. By [HR, Lemma 1], M is continuous. Hence M is a weakly injective continuous non-injective right R-module. □

On the other hand, weakly injective quasi-continuous modules do show some injective-like behaviour. While it is not true in general that direct summands of weakly K-injective modules is weakly K-injective, we observe the following

3.2. Theorem. *Let M be a weakly K-injective quasi-continuous module. Then every direct summand of M is weakly K-injective.*

Proof. Let $M = N \oplus T$ be a weakly K-injective quasi-continuous right R-module. Consider a map $\varphi : K \to E(N)$. Let $K_1 = \varphi(K) \cap N$. Notice that $K_1 \subset' \varphi(K)$. Since N is quasi-continuous, there exists a summand $K^* \subset^{\oplus} N$ with $K_1 \subset' K^*$. So, let $N = K^* \oplus H$. On the other hand, since M is weakly K-injective $\varphi(K) \subset M'$ for some submodule M' of $E(M)$ isomorphic to M. It follows by the quasi-continuity of M that $\varphi(K) \subset' K' \subset^{\oplus} M'$. Let $\varrho : M' \to M$ be an isomorphism. Then $\varrho(K') \subset^{\oplus} M$. Notice that $E(K^*) \cong E(\varrho(K'))$. It follows from the quasi-continuity of M that $K^* \cong \varrho(K') \cong K'$. The identity map $i : \varphi(K) \to \varphi(K)$ extends to an embedding $i^* : K' \to E(\varphi(K)) \subset E(N)$. Since $K_1 \subset' \varphi(K) \subset' i^*(K')$, we get $i^*(K') \cap H = 0$. Thus $\varphi(K) \subset i^*(K') \oplus H \cong K^* \oplus H = N$, proving that N is weakly K-injective. □

3.3. Corollary. *Every direct summand of a weakly injective quasi-continuous module is weakly injective.*

It seems natural next to study rings for which every weakly injective continuous module is injective. Proposition 3.1 illustrates that being noetherian is not sufficient to guarantee for this property to hold. The characterization of such rings appears to be quite complicated. We show next that no reasonable chain condition seems to be necessary either by presenting several examples of rings for which every weakly injective continuous module is injective.

3.4. Example. *Let R be any commutative noetherian ring. Then every weakly injective continuous module is injective.*

Proof. By [R], if R is commutative and noetherian, every continuous module is quasi-injective. Quasi-injective weakly injective modules are always injective. □

3.5. Example. *The ring*

$$R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a \in End_{\mathbf{Z}}(\mathbf{Z}(p^{\infty})), b \in \mathbf{Z}(p^{\infty})\}$$

is a non-noetherian ring for which every weakly injective continuous right module is injective.

Proof. As seen in Example 24.34 of [F], R is a commutative self-injective uniserial ring with

$$R/Z(p^{\infty}) \cong End_{\mathbf{Z}}(\mathbf{Z}(p^{\infty})),$$

and $End_{\mathbf{Z}}(\mathbf{Z}(p^{\infty}))$ is isomorphic to the ring of p-adic integers which is commutative and noetherian. Let M be a weakly-injective continuous R-module. Then $M = A \oplus B$ where A has essential socle and $soc(B) = 0$. By Corollary 3.3, A and B are weakly injective. It is easy to see that $\mathbf{Z}(p^{\infty}) \subset ann_R(B)$ and therefore B is a weakly injective continuous $R/\mathbf{Z}(p^{\infty})$-module. By [R], B is a quasi-injective $R/\mathbf{Z}(p^{\infty})$-module. It follows that B is an injective $R/\mathbf{Z}(p^{\infty})$-module and so B_R is quasi-injective. Since B_R is weakly injective, B is injective. Now, let A_1 be a uniform direct summand of A, say $A = A_1 \oplus A_2$. Then A_1 and A_2 are weakly injective and mutually injective. Since $E(A_1) \cong R_R$, we see that A_1 is isomorphic to an ideal of R. But no proper ideal of R_R may be weakly injective, therefore $A_1 \cong R_R$. Hence, A_1 and A_2 are injective. In conclusion, M is injective, as claimed. □

Rings whose cyclic right modules have finite uniform dimension (called right qfd rings) sometimes play a role similar to that of noetherian rings when studying weakly injective modules [AJL1]. For that reason, while being right noetherian is not necessary for all weakly injective continuous right modules to be injective one might expect that being right qfd is. Our next example rules out this implication.

3.6. Example. *Any commutative semiprimary ring with $J^2 = 0$ (where J is the Jacobson radical of R) has the property that every weakly injective continuous module is injective.*

Proof. If R is a commutative semiprimary ring with $J^2 = 0$ then every continuous module is quasi-injective [R]. Consequently, every weakly injective continuous R-module is injective. □

One could expect that the injective-like behaviour of weakly injective continuous modules might be to the extent that the direct sum of any two such modules would once again be continuous. That this is not the case is the purpose of our next proposition.

3.7. Proposition. *A ring R is such that every weakly injective continuous module is injective if and only if any finite direct sum of weakly injective continuous modules is continuous.*

Proof. Necessity is clear. Conversely, consider a weakly injective continuous module M. If M^2 were continuous then M would be quasi-injective and hence injective. □

Let us refer to a ring R as being right Σ-(weakly) injective if for any set $\mathcal{A}$, the R-module $R_R^{(\mathcal{A})}$ is (weakly) injective. By [F, Proposition 20.3A], a right Σ-injective ring is quasi-Frobenius.

3.8. Remark. *There is a commutative right Σ-weakly injective semiperfect ring which is not semiprimary and not noetherian.*

Proof. Let R be a ring of Example 3.5. Since R is right q.f.d., arbitrary direct sums of weakly injective modules are weakly injective [AJL1]. In particular, R is right Σ-weakly injective. Furthermore, we notice that $R^{(\omega)}$ is not continuous, indeed not even CS, because otherwise R would be quasi-Frobenius by [H, Corollary 2]. □

We conclude with a positive result.

3.9. Proposition. *A left perfect right Σ-weakly injective ring is quasi-Frobenius.*

Proof. Let R be left perfect and right Σ-weakly injective. By [AJL2, Theorem 3.3], R is indeed right self-injective. Consider any cyclic right R-module X. Since R is left perfect, $\text{soc}(X) \subset' X$ and there exists a set $\mathcal{A}$ such that $\text{soc}(X)$ embeds in $R^{(\mathcal{A})}$. Extend the embedding to one of X into $E(R^{(\mathcal{A})}$. Since $R^{(\mathcal{A})}$ is weakly injective we get that X indeed embeds in $R^{(\mathcal{A})}$ hence in R^n, for some $n \in \mathbf{Z}^+$. But then X has finite uniform dimension. Since the cyclic module X was chosen arbitrarily, we have shown that every cyclic module has an essential and finitely generated socle. It follows that R is right artinian (cf. [F, Corollary 19.16B]) and thus quasi-Frobenius. □

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