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ON A CLASS OF NON-NOETHERIAN V-RINGS

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Abstract. Right V-rings R with infinitely generated right socle $Soc(R_R)$ such that $R/Soc(R_R)$ is a division ring are characterized as those non-noetherian rings over which a cyclic right module is either non-singular or injective. Furthermore, it is shown that a non-noetherian, right V-ring S is Morita-equivalent to a ring of this type iff all singular simple right S-modules are isomorphic and every direct sum of uniform modules with an injective module over S is extending.

1. Introduction. A ring R is said to be of type (*) if R is a right V-ring with infinitely generated $Soc(R_R)$ such that $R/Soc(R_R)$ is a division ring. Rings of this type have been considered to show, among others, that:

a) there are right SI-rings which are not left SI ([7, Example 3.2]),

b) there are right V-rings which are not left V ([8, Example 6.19]), and

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c) there exist rings, not necessarily right noetherian, over which direct sums of uniform right modules and *an* injective right module are extending ([9]).

In view of the importance of these rings it would be interesting to investigate the behavior of modules over them, especially that of cyclic modules.

We will show that rings of type (*) are exactly rings which are not right noetherian and each of their cyclic right modules is either non-singular or injective. Moreover, a ring R is Morita-equivalent to a ring of type (*) if and only if R is a non-noetherian right V-ring such that R satisfies the module theoretic condition stated in c) and all singular simple right R-modules are isomorphic.

2. Preliminaries. Throughout this paper, all rings have identity and all modules are unitary right modules, unless otherwise stated. For a module M we denote by $Soc(M)$, $Z(M)$ and $J(M)$ the socle, the singular submodule and the Jacobson radical of M, respectively. If $M = Soc(M)$, we say that M is a semisimple module. A ring R is called a *semisimple* ring if R_R is semisimple. A module M is called a singular (resp., non-singular) module if $Z(M) = M$ (resp., $Z(M) = 0$). A ring R is called right non-singular if $Z(R_R) = 0$.

For general background and terminology we refer to Anderson-Fuller [1] and Faith [5].

A ring R is called a right (left) SI-ring if every singular right (left) R-module is injective. SI-rings have been introduced and investigated by Goodearl [7]. The following structure theorem is useful in our investigation.

Lemma 1 ([7, 3.11]). *A ring R is right SI if and only if R is right non-singular and $R = K \oplus R_1 \oplus \cdots \oplus R_n$, a ring direct sum, where $K/Soc(K_K)$ is semisimple and each R_i is Morita-equivalent to a right SI-domain D_i which is not a division ring.*

It is known that if D is a right SI-domain, then D is right noetherian, right hereditary and for each non-zero right ideal C of D, D/C is semisimple. By [10, Corollary 5], a ring R is right SI if and only if every cyclic singular right R-module is injective. This shows that right SI-domains are precisely the same as right PCI-domains introduced by Faith [4].

We record some other well-known results which will be refered to throughout the paper.

Lemma 2 ([10]). *If every cyclic singular R-module is injective, then R is a right SI-ring.*

Lemma 3 ([6], [11]). *A ring R is the direct sum of a semisimple ring and a right PCI-domain (equivalently, a right SI-domain) if and only if every cyclic R-module is projective or injective.*

We notice that the statement "R is a direct sum of a semisimple ring and a right SI-domain" means that R is either a semisimple ring, or a right SI-domain, or the direct sum of two such rings.

Lemma 4 ([2]). *Let M be a finitely generated quasi-injective module and $\{N_i\}$ be a system of infinitely many independent submodules of M whose sum is N. Then M/N has infinite uniform dimension.*

3. The results. For convenience we say that a ring R satisfies condition (E) if every cyclic R-module is non-singular or injective. Notice that homomorphic images of rings satisfying (E) also satisfy (E).

Lemma 5. *Let R be a semiprimary ring. If R satisfies (E), then R is semisimple.*

Proof. Let R be a semiprimary ring satisfying (E) and let $S = Soc(J(R))$. Then S is an $R/J(R)$-module. Write $R_R = e_1R \oplus \cdots \oplus e_nR$

where each e_i is a primitive idempotent of R. Now, if $J(R)$ is nonzero, then there is an e_i with $e_iJ(R) \neq 0$. Since e_iR is a local module with Jacobson radical $e_iJ(R)$, it is clear that $e_iR/e_iJ(R)$ is a singular simple R-module. Hence $R/J(R)$ can not be non-singular as an R-module. Thus, by (E), $R/J(R)$ must be an injective R-module, and so $J(R)$ contains an injective simple submodule, a contradiction. Hence $J(R) = 0$. □

Next we characterize rings satisfying condition (E).

Theorem 6. *For a ring R the following conditions are equivalent:*

(i) Every cyclic R-module is non-singular or injective;

(ii) R is either a ring direct sum of a semisimple ring and a right SI-domain or R is a ring of type (*).

Consequently, a non-noetherian ring satisfies (E) if and only if it is a ring of type (*).

Proof. $(i) \Rightarrow (ii)$. By (i), every cyclic singular R-module is injective. Hence by Lemma 2, R is right SI. Therefore by Lemma 1, $R = A \oplus B$, where $A/Soc(A_A)$ is semisimple and B is a semiprime right noetherian ring with zero right and (left) socle.

Case 1. $B \neq 0$.

If B_B is not uniform, there are finitely many uniform right ideals U_1, ..., U_m of B ($m \geq 2$) such that $U_1 \oplus \cdots \oplus U_m$ is essential in B_B. Let V_i be a non-zero proper submodule of U_i. Then B/V_i is not non-singular, and so injective by (E). Hence the injective hull of $U_1 \oplus \cdots \oplus U_{i-1} \oplus U_{i+1} \oplus \cdots \oplus U_m$ is cyclic for every $i = 1, 2, ..., m$. Therefore, the injective hull of the B-module B is finitely generated. Since B is right noetherian and semiprime, B is semisimple by [5, Theorem 20.12], a contradiction to $Soc(B_B) = 0$. Hence B_B is uniform, and so B is a right SI-domain.

Since $Soc(A_A)$ is essential in A, the R-module $A/Soc(A_A)$ is singular. Hence if $A \neq Soc(A_A)$, then the factor module $R/Soc(A_A)$ cannot be non-singular, and hence it is injective by (E). But in this case B is a right self-injective domain, consequently a division ring, a contradiction. Thus, we must have $A = Soc(A_A)$, proving that R is a direct sum of a semisimple ring and a right SI-domain.

Case 2. $B = 0$.

In this case R is a right SI-ring such that $R/Soc(R_R)$ is a semisimple ring. If $Soc(R_R)$ is finitely generated, then R is a right artinian ring, hence semisimple by Lemma 5, and so we are done.

It remains to consider the case when $Soc(R_R)$ is infinitely generated. Let $S = eR$ be a minimal right ideal of R where e is an idempotent, and $K = (1-e)R$. It follows that $K/Soc(K_R)$ is singular and nonzero. Hence, by (E), $R/Soc(K_R)$ must be injective, and so S_R is injective. Consequently, every idempotent minimal right ideal of R is injective.

Therefore, the sum I of all idempotent minimal right ideals of R is an (two-sided) ideal of R, and so R/I satisfies (E). Moreover, for any nilpotent right ideal N of R, $Soc(R_R).N = 0$. Hence $Soc(R_R)^2 \subseteq I$. This together with the fact that $R/Soc(R_R)$ is a semisimple ring shows that R/I is a semiprimary ring. Hence, by Lemma 5, R/I is semisimple. This shows that $J(R) = 0$. Therefore, every simple R-module is injective, i.e. R is a right V-ring. In particular, every finitely generated right ideal of R in $Soc(R_R)$ is a direct summand of R_R. This implies that $Soc(R_R)$ is a von Neumann regular ideal of R. Since $R/Soc(R_R)$ is semisimple, we conclude that R is a von Neumann regular ring (cf. [8, Lemma 1.3]).

Now, let $R/Soc(R_R) = S_1 \oplus \cdots \oplus S_t$, where each S_i is a minimal right ideal of $R/Soc(R_R)$. Take an element $x \in R$ such that $x + Soc(R_R)$ generates S_1. Then $Soc(xR)$ is infinitely generated and $xR/Soc(xR)$ is singular. On the other hand, since R is regular,

$$R_R = xR \oplus C$$

for some right ideal C of R. It follows that $R/Soc(xR) \cong (xR/Soc(xR)) \oplus C$, and so $R/Soc(xR)$ is not non-singular, hence it is injective by (E). In particular, C is an injective cyclic R-module. Clearly, $C/Soc(C)$ is semisimple and hence of finite composition length. By Lemma 4, $Soc(C)$ must be finitely generated. It yields $C = Soc(C)$, i.e. C is contained in $Soc(R_R)$, and so we have

$$R/Soc(R_R) \cong xR/Soc(xR).$$

Thus $R/Soc(R_R)$ is a division ring, proving the fact that R is a ring of type (*).

$(ii) \Rightarrow (i)$. If R is a direct sum of a semisimple ring and a right SI-domain, then R satisfies (E) by Lemma 3. Hence we need only consider the case when R is of type (*), i.e. R is a right V-ring with infinitely generated $Soc(R_R)$ such that $R/Soc(R_R)$ is a division ring. Obviously, R is right non-singular and hence a right SI-ring by Lemma 1.

Let N be a non-zero right ideal of R. If $N \not\subseteq Soc(R_R)$, then $N + Soc(R_R) = R$ and $(N + Soc(R_R))/Soc(R_R)$ is simple. Hence there is an element $y \in N$ such that $yR + Soc(R_R) = R$. From this, R/N is either zero or non-singular, as desired.

If $N \subseteq Soc(R_R)$, we consider the factor module $\overline{R} = R/N$. Since R is a right SI-ring,

$$\overline{R}_R = \overline{U} \oplus \overline{V},$$

where $\overline{U}_R$ is non-singular and $\overline{V}_R$ is singular and injective. Put $Soc(R_R) = N \oplus M$, and denote by $\overline{M}$ the image of M in $\overline{R}$. By assumption, $\overline{R}/\overline{M}$ is a simple module over R. It follows that

$$(\overline{V} \oplus \overline{M})/\overline{M}$$

is either zero or simple. (It is clear that $\overline{V} \cap \overline{M} = 0$.) If it is zero, then $\overline{V} = 0$ and so $\overline{R} = \overline{U}$ is non-singular. If it is simple, then $\overline{V}$ is simple and so

$\overline{V} \oplus \overline{M} = \overline{R}$. In particular $\overline{M}_R$ is finitely generated, i.e. $\overline{M}_R$ is a direct sum of finitely many (injective) simple R-modules. Thus $(R/N)_R$ is injective. □

A module M is called an *extending* (or CS-) module if every submodule of M is essential in a direct summand of M (or equivalently, if every complement submodule of M is a direct summand). Extending modules have been extensively studied by many authors. We refer to [3] for references on this subject.

Rings, for which direct sums of certain type of extending modules are extending, have been recently considered in [9]. It was shown there that, *for a ring R, every uniform R-module has composition length at most 2 if and only if every direct sum of uniform R-modules is extending.* In general, such a ring need not be right semi-artinian.

A ring R is said to satisfy condition (E') *if every direct sum of uniform R-modules and an injective R-module is extending.* Condition (E') has been considered first in [9] where the following result was obtained:

Lemma 7 ([9]). *A ring R satisfying (E') is right semi-artinian, i.e. every non-zero R-module has a non-zero socle. If R is right non-singular, then R satisfies (E') if and only if R is right SI and each non-singular uniform R-module has composition length at most 2.*

Now we consider some semiprime rings satisfying condition (E').

Theorem 8. *For a non-noetherian semiprime ring R consider the following conditions:*

(i) R satisfies (E') and all singular simple R-modules are isomorphic;

(ii) $R/Soc(R_R)$ is a simple artinian ring;

(iii) R is Morita-equivalent to a ring S such that $S/Soc(S_S)$ is a division ring.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. $(i) \Rightarrow (ii)$. By Lemmas 1 and 7, $R/Soc(R_R)$ is a semisimple ring. In particular, $R/Soc(R_R)$ is a direct sum of finitely many singular simple R-modules, which are isomorphic to each other by (i). Hence $R/Soc(R_R)$ is a simple artinian ring, proving (ii).

$(ii) \Rightarrow (iii)$. Since R is semiprime, (ii) gives that R is von Neumann regular. In particular, R is right non-singular. Hence, by Lemma 1, R is a right SI-ring. Using the same argument as that in the first part of the proof of Theorem 6 (the last part of Case 2) we can show that R_R has the following direct decomposition:

$$R_R = R_1 \oplus \cdots \oplus R_n, \tag{1}$$

where each $R_i/Soc(R_i)$ is simple and $R_i/Soc(R_i) \cong R_j/Soc(R_j)$ for each $i, j = 1, 2, ..., n$. Note that each $Soc(R_i)$ is infinitely generated, and since R is right SI, R is right hereditary (cf. [7]).

For each i, with $i = 2, 3, ..., n$, there exists a homomorhism φ_i' of R_1 onto $R_i/Soc(R_i)$. Let φ_i'' be the canonical homomorphism of R_i onto $R_i/Soc(R_i)$. By the projectivity of R_1, there is a homomorphism φ_i from R_1 to R_i such that $\varphi_i''.\varphi_i = \varphi_i'$. Hence $\varphi_i(R_1)$ is not semisimple, and so $R_i = \varphi_i(R_1) + Soc(R_i)$. Therefore there is a submodule C_i of R_i with $C_i \subset Soc(R_i)$ such that

$$R_i = \varphi_i(R_1) \oplus C_i.$$

Hence C_i is a direct sum of finitely many minimal submodules of R_i.

Moreover, since $\varphi_i(R_1)$ is projective, we have

$$R_1 \cong \varphi_i(R_1) \oplus Ker(\varphi_i).$$

Hence each R_i is isomorphic to a direct summand of $R_1 \oplus C_i$. Put

$$A = R_1 \oplus C_2 \oplus \cdots \oplus C_n.$$

Then A is a finitely generated right ideal of the regular ring R and $A/Soc(A)$ is a simple R-module. Write $A = eR$ where e is an idempotent. Moreover, if we denote by M the external direct sum of n copies of eR, then

$$M \cong R_R \oplus L$$

for some submodule L of M. This shows that eR is a projective generator of the category Mod-R, and hence R is Morita-equivalent to $End_R(eR) \cong eRe$. It is clear that eRe is a semiprime ring and $eRe/Soc(eRe)$ is a division ring. This proves (iii).

$(iii) \Rightarrow (ii)$ is clear. □

The following result is a consequence of Theorems 6 and 8, which also shows how the conditions (E) and (E') are related to each other.

Corollary 9. *For a non-noetherian ring R, the following conditions are equivalent:*

(i) R is a right V-ring satisfying (E') and all singular simple right R-modules are isomorphic;

(ii) R is a right V-ring and $R/Soc(R_R)$ is simple artinian;

(iii) R is Morita-equivalent to a ring satisfying (E), i.e. a ring of type (*).

Because we are concerned with the question of characterizing rings, Morita-equivalent to rings of type (*), we restricted ourself in Theorem 8 on those rings which are simple artinian modulo socle. In fact, Theorem 8 can be extended to a general form as discussed below.

Let R be a non-noetherian semiprime ring and assume that $R/Soc(R_R)$ is semisimple. Hence R is von Neumann regular. Let R_1 be a principal right ideal of R such that $(R_1 + Soc(R_R))/Soc(R_R)$ is simple. Then $R_R = R_1 \oplus R_1'$ and $Soc(R_1)$ is infinitely generated. Therefore, by an easy induction proof we obtain the following direct decomposition for R_R:

$$R_R = R_1 \oplus \cdots \oplus R_n, \tag{2}$$

where each $R_i/Soc(R_i)$ is simple and each $Soc(R_i)$ is infinitely generated. (Notice that in this case some factor modules $R_i/Soc(R_i)$ may not be isomorphic to each other.)

Moreover, by Lemma 1, R is right SI and hence right hereditary. Therefore, we may use the same argument as that in the proof of Theorem 8, to find finitely many *independent* principal right ideals of R, say $A_1, \ldots, A_t$, with the following properties:

1. Each $A_j/Soc(A_j)$ is simple,
2. For $j \neq k$, $A_j/Soc(A_j) \not\cong A_k/Soc(A_k)$,
3. Each R_i is isomorphic to a direct summand of exactly one A_j.

Let $A = A_1 \oplus \cdots \oplus A_t$ and as earlier let M be the external direct sum of n copies of A_R. Then we get $M \cong R_R \oplus C$ for some submodule C of M. Hence A is a generator of Mod-R. Since A is a finitely generated right ideal of R, $A = fR$ for some idempotent $f \in R$. Therefore, R is Morita-equivalent to the ring $End_R(fR) \cong fRf$. It is also easy to see that $fRf/Soc(fRf)$ is a direct sum of finitely many division rings.

If we assume moreover that all idempotents of R are central, then (2) is a ring-direct decomposition of R. Furthermore, in this case, R is a right V-ring. Then we conclude easily that each R_i in (2) is a ring of type (*).

Thus we have proved the following result.

Theorem 10. *For a non-noetherian semiprime ring R, consider the following conditions:*

(i) R satisfies (E');

(ii) $R/Soc(R_R)$ is a semisimple ring;

(iii) R is Morita-equivalent to a ring T such that $T/Soc(T_T)$ is a direct sum of finitely many division rings.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). In addition, if all idempotents of R are central, then (i) and (ii) are equivalent to the condition:

(iii') R is a ring-direct sum of finitely many rings satisfying (E), i.e. rings of type (*).

In connection with condition (E') we would like to mention a result obtained in [9]:

For a right non-singular ring R, the following conditions are equivalent:

(a) Every direct sum of injective R-modules and uniform R-modules is extending;

(b) Every direct sum of extending R-modules is extending;

(c) R is right artinian, and every uniform R-module has length at most 2.

In a revised version, the authors of [9] have shown furthermore that, under this condition (that R is right non-singular), $(a) - (c)$ are equivalent to:

(d) Every R-module is extending.

It has also been shown in [9], that in general, a ring satisfying condition (b) is semiprimary with Jacobson radical square zero.

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REFERENCES

1. F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer-Verlag (1992).

5. C. Faith, Algebra II: Ring Theory, Springer-Verlag (1976).

6. V.K. Goel, S.K. Jain and S. Singh, Rings whose cyclic modules are injective or projective, Proc. Amer. Math. Soc. 53 (1975), 16-18.

7. K.R. Goodearl, Singular torsion and splitting properties, Memoirs of the Amer. Math. Soc. 124 (1972).

8. K.R. Goodearl, Von Neumann Regular Rings, Pitman, London (1979).

9. D.V. Huynh and B.J. Müller, Rings for which direct sums of extending modules are extending, preprint, (April 1994); announcement without proofs in *Vietnam Journal of Mathematics*, Vol. 22 (No. 3 & 4) (1994), 120-123.

10. B.L. Osofsky and P. F. Smith, Cyclic modules whose quotients have complements direct summands, J. Algebra 139 (1991), 342-354.

11. P. F. Smith, Rings characterized by their cyclic modules, Canad. J. Math. 24 (1979), 93-111.

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