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When Is a Simple Ring Noetherian?

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A module M is known to be a CS-module (or an extending module) if every complement submodule of M is a direct summand. It is shown that (i) a simple ring R must be right noetherian if every cyclic singular right R-module is CS, and (ii) over a simple ring R if every proper cyclic right module is quasi-injective, then R is right hereditary and right noetherian. © 1996 Academic Press, Inc.

1. INTRODUCTION

Simple noetherian rings are topics of considerable interest in ring theory and have been extensively studied by many authors (see, e.g., Chatters and Hajarnavis [1], Cozzens and Faith [3], Faith [9], and references therein). In this paper we consider the question, when is a simple ring noetherian, and prove the following Theorems A and B. By a *proper* cyclic right R-module we mean a cyclic right R-module that is not isomorphic to R_R.

THEOREM A. *If R is a simple ring such that every cyclic singular right R-module is CS, then R is right noetherian.*

From a theorem of Osofsky and Smith [16] it follows that if every cyclic right module over a ring R is CS then every cyclic right R-module has finite uniform dimension. However, in general such a ring need not be right noetherian. Theorem A shows that this is the case if R is simple. It would be interesting to consider the question whether or not a simple right noetherian ring R in Theorem A is always Morita equivalent to a domain.

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By a theorem of Faith [7], this is true if R contains a projective uniform right ideal.

THEOREM B. *Let R be a simple ring. If every proper cyclic right R-module is quasi-injective, then R is either artinian or a right noetherian right hereditary domain. Moreover, in this case, every proper cyclic right R-module is injective.*

Rings over which proper cyclics are injective (called PCI-rings) or singular cyclics are injective (called SI-rings) have been studied by many authors, including J. H. Cozzens, R. F. Damiano, C. Faith, K. R. Goodearl, B. L. Osofsky, P. F. Smith, and others. The right noetherian domains in Theorem B are exactly the right PCI-domains [4, 8], or equivalently, the right SI-domains [10]. The existence of PCI-domains has been shown in [2]. A right PCI-domain D has the property that for each non-zero right ideal A of D, D/A is semisimple. It is still unknown whether or not any right PCI-domain is left PCI (see, e.g., [16]). It is known that a right PCI-domain D is left PCI if and only if the direct sum $D_D \oplus D_D$ is a CS-module (cf. [6, 12.9]).

In Section 2 we study the structure of prime rings over which every proper cyclic right module is quasi-injective and show that such a ring is either simple artinian or a right Ore domain. This result gives a crucial step toward the proof of Theorem B. The proofs of Theorems A and B are presented in Section 3. As a consequence of Theorem B we obtain that a right V-ring R is either semisimple artinian or a right PCI-domain if every proper cyclic right R-module is quasi-injective.

Throughout we consider associative rings with identity and all modules are unitary. For a module M we denote by $\mathrm{soc}(M)$ and $E(M)$ the socle and the injective hull of M, respectively. If $M = \mathrm{soc}(M)$, then M is called a semisimple module. If M has finite composition length, then we denote its length by $l(M)$. For other terminology and notation not defined here we refer to Faith [9].

2. PRELIMINARY RESULTS

A ring R is called a right PCQI-ring if every proper cyclic right R-module is quasi-injective. It is known that a right PCQI-ring is either semiperfect or prime. Non-prime right PCQI-rings have been characterized in [12]. In this section we consider prime right PCQI-rings and show that such a ring is either artinian or a right Ore domain.

LEMMA 2.1. *Let R be a right PCQI-ring. Then any cyclic right R-module has finite uniform dimension.*

Proof. Let X be a cyclic right R-module and let E be an essential submodule of X. Then, clearly, $X/E \not\simeq R_R$. Moreover, we easily see that any cyclic subfactor of X/E is not isomorphic to R_R and thus quasi-injective. Hence, by [16, Theorem 1], X/E has finite uniform dimension.

Our aim now is to show that X/S has finite uniform dimension, where $S = \mathrm{soc}(X)$. Let T be a submodule of X such that $S \oplus T$ is essential in X. Hence, $X/(S \oplus T)$ and therefore $(X/S)/\overline{T}$, where $\overline{T} = (S \oplus T)/S$, has finite uniform dimension. From this fact and since $T \simeq \overline{T}$, we need only to show that T has finite uniform dimension. Assume, on the contrary, that T contains an essential submodule $V = \bigoplus_{i=1}^{\infty} V_i$, an infinite direct sum of non-zero submodules. Since $\mathrm{soc}(T) = 0$, every V_i contains an essential proper submodule W_i, and so $W = \bigoplus_{i=1}^{\infty} W_i$ is an essential submodule of V. Furthermore, because $S \oplus W$ is an essential submodule of X, $X/(S \oplus W)$ has finite uniform dimension. On the other hand, $X/(S \oplus W)$ contains

$$(S \oplus V)/(S \oplus W) \simeq V/W \simeq \bigoplus_{i=1}^{\infty} (V_i/W_i),$$

where each V_i/W_i is non-zero, a contradiction. Thus, T has finite uniform dimension.

To finish the proof, it suffices to show that $\mathrm{soc}(X)$ is finitely generated. Assume on the contrary that $\mathrm{soc}(X)$ is infinitely generated. Then we may write $\mathrm{soc}(X) = H \oplus K$, where H and K are infinitely generated. Since H cannot be a direct summand of X, $X/H \not\simeq R_R$. By hypothesis, X/H is a cyclic quasi-injective module. Let $\overline{K}$ be the image of K in X/H. Then $X/\mathrm{soc}(X) \simeq (X/H)/\overline{K}$. Hence $(X/H)/\overline{K}$ has finite uniform dimension. On the other hand, $\overline{K}$, being isomorphic to K, is a direct sum of infinitely many simple modules. By [5, Lemma 1], $(X/H)/\overline{K}$ cannot have finite uniform dimension, a contradiction. Thus, $\mathrm{soc}(X)$ has to be finitely generated, as desired. ∎

THEOREM 2.2. *A prime right PCQI-ring is either artinian or a right Ore domain.*

Proof. Let R be a prime right PCQI-ring. First we show that R is right non-singular. Assume, on the contrary, that the right singular ideal $Z(R_R)$ is non-zero. By Lemma 2.1, R has finite right uniform dimension and, therefore, there is a uniform submodule U of R_R. Since R is prime, $V = U \cap Z(R_R) \neq 0$ and $V^2 \neq 0$. Let $a \in V$ be such that $aV \neq 0$ and take $0 \neq x \in V \cap \mathrm{ann}_r(a)$. Since $x \in Z(R_R)$, then $xR \not\simeq R_R$; and therefore xR is quasi-injective. Let $E(V)$ be the injective hull of V. Then xR is a fully invariant submodule of $E(V)$. In particular, $V(xR) \subseteq xR$. Thus $(aV)(xR) = a(VxR) \subseteq a(xR) = 0$, while $aV \neq 0$, $xR \neq 0$, which is a contradiction to the primeness of R. Therefore, we must have $Z(R_R) = 0$, as desired.

Next, by Lemma 2.1, R has finite right uniform dimension. Since R is right non-singular, R is a right Goldie ring. If R_R is uniform, then R is either a division ring or a right Ore domain. Assume that R_R is not uniform. Let $U_1 \oplus \cdots \oplus U_m$ $(m > 1)$ be a direct sum of uniform right ideals which is an essential submodule of R_R. Let $0 \neq a_1 \in U_1$. Then $a_1 R \not\simeq R_R$. Therefore, $a_1 R$ is quasi-injective. Since R is a prime right Goldie ring, all uniform right ideals are subisomorphic to each other. Hence, each U_i $(i \geq 2)$ contains an isomorphic copy $a_i R$ of $a_1 R$. It follows that $A = a_1 R \oplus \cdots \oplus a_m R$ is quasi-injective. Since A is essential in R_R, A contains a regular element c. Thus, as A is cR-injective and $cR \simeq R$, A is injective. Therefore $A = R$ and R is right self-injective. It follows that R is simple artinian. ∎

Notice that, although a right PCI-domain is always simple right noetherian and right hereditary, this need not be true for right PCQI-domains, as the following example of Levy [13] shows.

EXAMPLE 2.3. *There exists a commutative valuation PCQI-domain which is neither noetherian nor hereditary.*

Proof. Let F be a field and let x be an indeterminate. The right of *formal power series* $R = \{\sum_{i \in W} a_i x^i | a_i \in F$ and W is a well-ordered subset of $\mathbb{R}^+ \cup \{0\}\}$ is a commutative valuation domain which is not noetherian, all of whose proper homomorphic images are self-injective [13]. Thus, R is a PCQI-ring. The fact that R is not hereditary follows from [1, Corollary 8.25]. ∎

3. THE PROOFS OF THEOREMS A AND B

The first part of the following lemma is essentially due to J. T. Stafford and it can be proved by induction on the composition length of bR as presented in [1, Theorem 14.1].

LEMMA 3.1. *Let R be a simple right Goldie ring which is not artinian. Let M be a singular right R-module. Assume that $M = aR \oplus bR$ for some $a, b \in M$ such that bR has finite composition length. Then $M = (a + bx)R$, for some $x \in R$.*

Consequently, if R is a simple right PCQI-domain, then every finitely generated artinian right R-module A is semisimple.

Proof. We need only to verify the second part of the lemma. We may assume that R is not a division ring. Clearly, A_R is singular and hence each cyclic submodule of A is proper. If A_R is not semisimple, then there exists a cyclic submodule X of A_R which is not semisimple. Let $\mathrm{soc}(X) = S_1 \oplus \cdots \oplus S_k$, where each S_i is simple. Without loss of generality, assume S_1 is not a direct summand of X. Then, by the first part of this lemma, $S_1 \oplus X$ is cyclic and hence, by hypothesis, quasi-injective. It follows that S_1 is X-injective and therefore it splits in X, a contradiction. Thus, A_R is semisimple, as claimed. ▮

Proof of Theorem A. Let R be a simple ring. If $\mathrm{soc}(R_R) \neq 0$, then $R = \mathrm{soc}(R_R)$, proving that R is simple artinian. Hence we consider the case $\mathrm{soc}(R_R) = 0$. Moreover, since R is simple, R is right non-singular.

Now we assume that every cyclic singular right R-module is CS. Let E be an essential right ideal of R. Then the factor module R/E is singular. Hence, by hypothesis, every cyclic subfactor of R/E is CS. By [16, Theorem 1], R/E has finite uniform dimension. By the same argument as in the proof of Lemma 2.1 we obtain that $R/\mathrm{soc}(R_R)$ has finite right uniform dimension, and so R has finite right uniform dimension. Thus R is a right Goldie ring. Moreover, let A be an arbitrary right ideal of R. Then there is a right ideal B of R such that $A \cap B = 0$ and $A \oplus B$ is an essential right ideal of R. As A, B, and $R/(A \oplus B)$ have finite uniform dimension, we conclude that R/A has finite uniform dimension, i.e., every cyclic right R-module has finite uniform dimension.

Again let E be an essential right ideal of R and let $M = R/E$. We aim to show first that M is noetherian. Let α be an ordinal. Then the socle series of M is defined inductively as

$$S_1 = \mathrm{soc}(M), \qquad S_\alpha/S_{\alpha-1} = \mathrm{soc}(M/S_{\alpha-1}),$$

and

$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta$$

if α is a limit ordinal. Let $S = \bigcup_\alpha S_\alpha$. Then the socle of $U = M/S$ is zero. We claim that U is noetherian. Assume that $U \neq 0$. Since U is CS and has finite uniform dimension, U is a direct sum of finitely many uniform submodules. Hence, without loss of generality, we may assume that U is uniform.

Let V be an arbitrary singular simple right R-module. Then, since R is a simple right Goldie ring, we may use Lemma 3.1 to see that the external direct sum $T = V \oplus U$ is cyclic. Hence by hypothesis, T is CS. For convenience of further observation we may consider U and V as submodules of T. It is clear that $\mathrm{soc}(T) = V$, a complement submodule of T. Let A be a submodule of U and assume that there is a non-zero homomorphism φ of A to V. Let $W = \{a - \varphi(a) | a \in A\}$. Then W is contained as an essential submodule in a direct summand W^* of T. Write $T = W^* \oplus B$ for some submodule B of T. Since $W^* \cap V = 0$, and $V = \mathrm{soc}(T)$, it follows that $V \subseteq B$. But, since V is a complement submodule of T and B is uniform, we must have $V = B$. Thus

$$T = W^* \oplus V.$$

Let π be the projection of T onto V along this decomposition. Then it is easy to check that $(\pi|U)$ is an extension of φ from U to V. This shows that V is U-injective, and hence every singular simple right R-module is U-injective.

If U is not noetherian, then there is an infinite strictly ascending chain of submodules in U

$$x_1R \subset x_1R + x_2R \subset \cdots.$$

Let $X = \bigcup_{i=1}^{\infty}(x_1R + \cdots + x_iR)$. Then, since every singular simple right R-module is X-injective, by a standard argument we can find a submodule Y of X such that X/Y is a direct sum of infinitely many simple modules. This is a contradiction to the fact that U/Y has finite uniform dimension. Thus U is noetherian, as claimed.

To show that M is noetherian, it is now enough to verify that S has finite composition length. Since every cyclic right R-module has finite uniform dimension, S_1 and each $S_{\alpha+1}/S_\alpha$ have finite composition length. Assume that $S_2 \neq S_3$. Then there exists $y \in S_3$ such that $yR \not\subseteq S_2$ and $(yR + S_2)/S_2$ is simple. Since yR is CS,

$$yR = K_1 \oplus \cdots \oplus K_m,$$

where each K_i is uniform. There is some K_i, say K_1, with $K_1 \not\subseteq S_2$. Again, since $K_1/\mathrm{soc}(K_1)$ is CS, there are finitely many non-zero submodules of K_1, say $H_1, \dots, H_n$, such that

$$K_1/\mathrm{soc}(K_1) = (H_1/\mathrm{soc}(K_1)) \oplus \cdots \oplus (H_n/\mathrm{soc}(K_1)),$$

where each $H_j/\mathrm{soc}(K_1)$ is simple or uniform and of composition length 2. Surely, there is some H_j, say H_1, such that $l(H_1/\mathrm{soc}(K_1)) = 2$. Then H_1 is a cyclic uniserial module with the unique composition series $\mathrm{soc}(K_1) \subset H \subset H_1$. Moreover, it is easy to verify that $H_1 \oplus (H/\mathrm{soc}(K_1))$ is not CS (cf. [11, Theorem 10] or [15, Lemma B]). However, by Lemma 3.1, $H_1 \oplus (H/\mathrm{soc}(K_1))$ is a cyclic singular right R-module and hence CS by hypothesis. This contradiction shows that $S_3 = S_2$ and so $S = S_2$. Thus S has finite composition length, as desired.

We have shown that for each essential right ideal E of R, R/E is noetherian. Hence $R/\mathrm{soc}(R_R)$ is right noetherian by [5]. But $\mathrm{soc}(R_R) = 0$, so R is right noetherian. ∎

The valuation commutative ring R in Example 2.3 has the property that every cyclic R-module is uniform and hence CS. But this ring is not noetherian. Thus the assumption in Theorem A that R is simple cannot be removed.

Since a quasi-injective module is CS, from Theorem 2.2 and Theorem A it follows

COROLLARY 3.2. *A simple right PCQI-ring is right noetherian.*

Recall that a ring R is said to satisfy the restricted minimum condition (RMC) if for each essential right ideal I of R, R/I is artinian.

Proof of Theorem B. Let R be a simple right PCQI-ring. Then by Theorem 2.2, R is either simple artinian or a right Ore domain. Hence we consider the latter case. It suffices to show that R satisfies the right RMC. For, if R satisfies the right RMC and I is a non-zero right ideal of R, the cyclic right module $X = R/I$ is artinian and hence semisimple by Lemma 3.1. So, R/I is injective by [10, Proposition 3.1], proving that R is right PCI.

Assume R does not satisfy the right RMC. Let A be a non-zero right ideal maximal with respect to the property that $M = R/A$ is not artinian. Note that, since R_R is uniform and $A \neq 0$, M_R is singular. The existence of A is guaranteed since, by Corollary 3.2, R is right noetherian. Then every proper factor module of M is artinian and hence semisimple (Lemma 3.1). Therefore, $\operatorname{soc}(M) = 0$, which implies that M is uniform. Let $\varphi \in \operatorname{End}_R(E(M))$. By hypothesis, every cyclic submodule of M is quasi-injective. Indeed, every submodule of M is quasi-injective. For, let U be any submodule of M. For any $x \in U$, $\varphi(x) \in xR \subseteq U$; therefore $\varphi(U) \subseteq U$, i.e., U is quasi-injective. (Here and below we use the known fact that a right R-module N is quasi-injective if and only if for each $\varphi \in \operatorname{End}_R(E(N))$, $\varphi(N) \subseteq N$; see, e.g., [9, p. 173].) Let M_1 be a maximal submodule of M. We will show that the external direct sum $T = M_1 \oplus M$ is quasi-injective. This would imply that M_1 is M-injective, but then M_1 would split in M, a contradiction to the uniformity of M. So, R must satisfy the right RMC.

Let $f \in \operatorname{End}_R(E(T))$. We aim to show that $f(T) \subseteq T$. This proves the quasi-injectivity of T. Clearly, $f(M_1 \oplus M_1) \subseteq M_1 \oplus M_1$, since $M_1 \oplus M_1$ is quasi-injective. Let M_2 be a maximal submodule of M_1. Then

$$l(T/(M_2 \oplus M_2)) = 3,$$

since $T/(M_2 \oplus M_2) \simeq (M_1/M_2) \oplus (M/M_2)$. Moreover, by Lemma 3.1, $T/(M_2 \oplus M_2)$ is cyclic. Therefore, there exists $d \in T$ such that

$$T = dR + (M_2 \oplus M_2), \qquad (*)$$

and so

$$l[dR/(dR \cap (M_2 \oplus M_2))] = 3. \qquad (**)$$

Assume that dR is uniform. Then we need to consider the following two cases:

Case 1. $dR \cap (0 \oplus M_2) = 0$.

Then dR is embeddable in M_1. If $V = dR \cap (M_2 \oplus 0) \neq 0$, then dR/V is artinian and hence semisimple by Lemma 3.1. It follows that dR/V is embedded in $\mathrm{soc}((M_1 \oplus M)/(M_2 \oplus 0))$. Hence, dR/V is simple since $(M_1 \oplus M)/(M_2 \oplus 0) \simeq (M_1/M_2) \oplus M$. Moreover, $V \subseteq W = dR \cap (M_2 \oplus M_2)$, and so dR/W must be simple. This is clearly a contradiction to $(**)$.

If $dR \cap (M_2 \oplus 0) = 0$, then

$$(M_1 \oplus 0) \oplus dR = (M_1 \oplus 0) \oplus D, \qquad (***)$$

where $dR \simeq D = [(M_1 \oplus 0) \oplus dR] \cap (0 \oplus M)$. If $D \not\subseteq (0 \oplus M_1)$, then there exists $d_1 \in M \setminus M_1$ such that $D \simeq (0 \oplus d_1 R)$. Since dR is embedded in M_1 there exists $d_2 \in M_1$ such that $d_1 R \simeq dR \simeq d_2 R$. But since M is uniform and $d_2 R$ is a quasi-injective submodule of M_1, we must have $d_1 R \subseteq d_2 R \subseteq M_1$, a contradiction. Thus $D \subseteq (0 \oplus M_1)$. From this and $(***)$ it follows that $dR + (M_1 \oplus M_1) \subseteq M_1 \oplus M_1$, a contradiction to $(**)$.

Case 2. $L = dR \cap (0 \oplus M_2) \neq 0$.

In this case, $dR \cap (M_1 \oplus 0) = 0$ and therefore dR is embedded in M. Hence dR/L is artinian and therefore semisimple by Lemma 3.1. It follows that dR/L is embedded in $\mathrm{soc}((M_1 \oplus M)/(0 \oplus M_2))$. Since $(M_1 \oplus M)/(0 \oplus M_2) \simeq M_1 \oplus (M/M_2)$, dR/L has length at most 2. Since $L \subseteq L' = dR \cap (M_2 \oplus M_2)$, $l(dR/L') \leq 2$. This yields a contradiction to $(**)$.

Thus dR is not uniform. Hence dR is essential in T. But since dR is quasi-injective, we must have $f(dR) \subseteq dR$. This and $(*)$ shows that $f(T) \subseteq T$, as desired. ∎

Note that the ring of integers is noetherian and PCQI but it is not PCI. Hence the assumption that R is simple cannot be removed.

Recall that a ring R is a right V-ring if every simple right R-module is injective. Clearly, a right PCI-ring is right V. This need not be the case for right PCQI-rings, in general (see Example 2.3).

COROLLARY 3.3. *Let R be a ring. Then R is right PCI if and only if R is right PCQI and right V.*

Proof. By [8], any right PCI-ring is right V and of course right PCQI.

Conversely, let R be a right PCQI- right V-ring. Since as a right V-ring R has zero Jacobson radical, R is either semisimple artinian or prime by [12]. Hence by Theorem 2.2, R is either semisimple artinian or a right Ore domain. If R is a right Ore domain, then R is a simple ring by [14, Lemma 3.1]. By Theorem B, R is a right PCI-ring. ∎

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REFERENCES

1. A. W. Chatters and C. R. Hajarnavis, "Rings with Chain Conditions," Pitman, London, 1980.
2. J. H. Cozzens, Homological properties of the ring of differential polynomials, *Bull. Amer. Math. Soc. (N.S.)* **76** (1970), 75–79.
3. J. H. Cozzens and C. Faith, "Simple Noetherian Rings," Cambridge Univ. Press, Cambridge, UK, 1975.
4. R. F. Damiano, A right PCI ring is right Noetherian, *Proc. Amer. Math. Soc.* **77** (1979), 11–14.
5. N. V. Dung, D. V. Huynh, and R. Wisbauer, Quasi-injective modules with ACC or DCC on essential submodules, *Arch. Math* **53** (1989), 252–255.
6. N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, "Extending Modules," Pitman, London, 1994.
7. C. Faith, A general Wedderburn theorem, *Bull. Amer. Math. Soc. (N.S.)* **73** (1967), 65–67.
8. C. Faith, When are all proper cyclics injective?, *Pacific J. Math.* **45** (1973), 97–112.
9. C. Faith, "Algebra I: Rings, Modules and Categories," Springer-Verlag, Berlin/New York, 1981.
10. K. R. Goodearl, Singular torsion and the splitting properties, *Mem. Amer. Math. Soc.* **124** (1972).
11. M. Harada and K. Oshiro, On extending property on direct sums of uniform modules, *Osaka J. Math.* **18** (1981), 767–785.
12. S. K. Jain, S. Singh, and R. G. Symonds, Rings whose proper cyclic modules are quasi-injective, *Pacific J. Math.* **67** (1976), 461–472.
13. L. S. Levy, Commutative rings whose proper homomorphic images are self-injective, *Pacific J. Math.* **18** (1966), 149–153.
14. G. O. Michler and O. E. Villamayor, On rings whose simple modules are injective, *J. Algebra* **25** (1973), 185–201.
15. B. L. Osofsky, Injective modules over twisted polynomial rings, *Nagoya Math. J.* **119** (1990), 107–114.
16. B. Osofsky and P. F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, *J. Algebra* **139** (1991), 342–354.