COMMUNICATIONS IN ALGEBRA, 23(5), 1663–1677 (1995)

Superfluous covers

S.K. Jain* and Sergio R. López-Permouth†
Ohio University
Athens, Ohio 45701

Louis H. Rowen
Bar-Ilan University
Ramat-Gan 52900, Israel

Throughout, M denotes a module over a ring R. Recall that a projective cover of M is an epic $\pi: P \to M$ with P projective and $\ker \pi$ small in P (to be defined below). Although projective covers have earned their important role as the dual of the injective hull, they have one major disadvantage — they need not exist. This difficulty motivates one to drop "projective" from the term "projective cover"; and the object of this paper is to study what can be said in this more general situation. Accordingly, we recall the following definition.

DEFINITION 1: A submodule K of M is small or superfluous, if $K+N < M$ for all $N < M$ (where $<$ means " proper submodule"). We denote the relation "K is a small submodule of M "by $K \ll M$. A superfluous cover is an epic $\pi: M' \to M$ with $\ker \pi$ small in M'; the cover is proper if $\ker \pi \neq 0$. M will be called auto-covering if it has no proper superfluous cover. A superfluous cover $\pi: M' \to M$ is called a projective (resp. flat) cover if M' is projective (resp. flat). A superfluous cover $\pi: M' \to M$ is maximal if M' is auto-covering.

It is well-known that projective modules are auto-covering. Thus, any projective cover is maximal.

Among other questions, we shall address the following:

1. Need every M have a flat cover?

* Partially supported by NSF grant # 9210491. † Partially supported by the Ministry of Education and Science of the Government of Spain and the Centre de Recerca Matemàtica (Barcelona)

2. A flat module with a projective cover is itself projective and thus is autocovering. Is it true in general that flat modules are autocovering?
3. Do maximal superfluous covers necessarily exist for an arbitrary module?
4. For which rings do all (f.g.) modules have maximal superfluous covers?
5. Which modules are auto-covering?
6. What properties do autocovering modules share with projective modules?

The answer to questions 1,2 and 3 is "No." Concerning question 4, our main result, Theorem 3.7, is that for many semilocal rings, a maximal superfluous cover must be a projective cover. Thus, while in many familiar cases maximal superfluous covers provide the full theory of projective covers, they also yield a theory in those cases where projective covers fail to exist. Regarding question 5, we shall study superfluous covers of cyclic modules. Intrinsic characterizations of auto-covering cyclic modules are given in Example 1.6 and throughout §2. In particular, we show that cyclic flat modules over commutative rings are auto-covering. Also, in §4 we study superfluous covers of modules over a Dedekind domain. We see that injective modules play a key role in that study. We show that every torsion free module over a Dedekind domain is auto-covering. Concerning question 6, our results include the fact that finite direct sums of autocovering modules are autocovering (Lemma 3.1.).

§1. Basic results about superfluous covers.

Let us start by collecting facts about small submodules. Write $K \ll M$ to denote K is small in M. Basic references include Anderson-Fuller [**1**] and Rowen [**8**]. Given a module M, define $\mathrm{Rad}(M)$ to be the intersection of all proper maximal submodules of M. (We take $\mathrm{Rad}(M) = M$ if M has no proper maximal submodules.) Then $\mathrm{Rad}(M)$ is also the sum of all small submodules of M; however, $\mathrm{Rad}(M)$ itself need not be small, in general, but is small when M is finitely generated. We write "f.g." for "finitely generated," and "Jac" for the Jacobson radical of a ring. We recall some well known facts about small submodules.

REMARK 1.0: (i) If $K \ll N$ and N is a submodule of M then $K \ll M$. (Indeed if $K + L = M$ then $K + (L \cap N) = N$, implying $N = L \cap N$, so $K \subseteq L \cap N \subseteq L$, and $L = M$.)

(ii) If P is a projective R-module then $\mathrm{Rad}(P) = \mathrm{Jac}(R)P$.

REMARK 1.1.

Then N/K is small in M/K iff for any submodule N' of M for which $N + N' = M$ one also has $K + N' = M$.

REMARK 1.2: Let $K < N < M$. If N/K is small in M/K then N is contained in the intersection $\tilde{N}$ of those maximal submodules of M which contain K. (Indeed, $\tilde{N}/K = \text{Rad}(M/K)$.)

REMARK 1.3: The following trick enables one to pass from cyclic modules to f.g. modules: If N is an R-module generated by n elements then $N^{(n)}$ is a cyclic $M_n(R)$-module; furthermore, $K \ll N$ as R-modules iff $K^{(n)} \ll N^{(n)}$ as $M_n(R)$-modules.

EXAMPLE 1.4: Let us consider the special case when $M = R$ and K is a left ideal. Let $\sqrt{K}$ denote the intersection of all maximal left ideals containing K. By Remark 1.2, N/K is small in R/K iff $N \subseteq \sqrt{K}$. Furthermore, mimicking the usual Jacobson theory, one has the following element-wise criterion: $a \in \sqrt{K}$ iff for every r in R there is s in R such that $s(1 - ra) - 1 \in K$, i.e., "ra is quasi invertible modulo K."

REMARK 1.5: If $M = \sum_{i \in I} Ra_i$ and $\pi\colon M' \to M$ is a superfluous cover then taking a set $\{b_i : i \in I\} \subset M'$ such that $\pi(b_i) = a_i$ for each i, we have $M' = \sum_{i \in I} Rb_i$. (Indeed, $\sum Rb_i + \ker \pi = M'$, so $\sum Rb_i = M'$.)

EXAMPLE 1.6: We shall now describe all superfluous covers for cyclic modules. Suppose $M = R/L$, and $\pi\colon M' \to M$ is a superfluous cover. By remark 1.5, M' is cyclic, so $M' \approx R/L'$ for some $L' < L$. Then $L/L' = \ker \pi \ll R/L'$, which by Example 1.4 is the case iff $L \subseteq \sqrt{L'}$, i.e., iff $\sqrt{L} = \sqrt{L'}$.

For example, taking $L' = L^2$ we have $\sqrt{L} = \sqrt{L'}$,since every maximal left ideal is prime. So, R/L^2 is necessarily a superfluous cover of R/L. Of course, this is proper only when $L \neq L^2$, so we have

REMARK 1.7: If R/L is auto-covering, then $L = L^2$, i.e., L is idempotent.

More generally, one gets:

REMARK 1.8: Suppose R/K is a superfluous cover of R/L. Then R/LK is a superfluous cover of R/K, and of R/L. (Indeed, if LK is contained in a maximal left ideal L' of R then either $K \subseteq L'$ or $L \subseteq L'$, implying $K \subseteq L'$ and $L \subseteq L'$.)

Let us answer question 1 negatively, by analyzing what happens with V-rings. Recall [7] that R is a left V-ring iff every module has radical 0, iff every cyclic module has radical 0.

PROPOSITION 1.9. *The following are equivalent:*

(1) *R is a left V-ring.*

(2) *Every cyclic R-module is auto-covering.*

(3) *Every f.g. R-module is auto covering.*

PROOF: (1) $\Rightarrow$ (3) If $\pi: M' \to M$ is a superfluous cover then $\ker \pi << M'$ so $\ker \pi = 0$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) If $K \ll M$ for M cyclic then $M \to M/K$ is a superfluous cover, implying $K = 0$. ∎

On the other hand, a ring R is von Neumann regular iff R/L is flat for each $L < R$, so taking a V-ring which is not von Neumann regular, one has a non-flat auto-covering cyclic module. Note this example must be noncommutative. In fact it cannot be a ring with polynomial identity, cf. [**2**]. In the next example we see that a module can have nontrivial covers but yet have no flat cover.

EXAMPLE 1.10: If R is Noetherian then every f.g. flat module is projective; thus, a cyclic module without a projective cover also fails to have a flat cover. For example, if R is a principal ideal domain (PID) then the superfluous covers of the cyclic module R/Ra are precisely R/Rb where $a|b$ and where b has the same prime divisors as a (so that $\sqrt{Ra} = \sqrt{Rb}$). But R/Rb cannot be projective for $b \neq 0$, since otherwise Rb is a summand of R, which is impossible because R has no nontrivial idempotents.

Furthermore, since R/Ra^2 is always a superfluous cover of R/Ra, we see that no proper cyclic module of a PID is auto covering. (Note this fails for principal left ideal domains, since in fact there are PLID's which are V-rings (cf. [**3**]).

REMARK 1.11: A finitely generated module M is a superfluous cover of a simple module iff $\mathrm{Rad}(M)$ is maximal. (Indeed, if $\pi: M \to S$ is a superfluous cover, then $\ker \pi \leq \mathrm{Rad}\, M$, implying $\mathrm{Rad}\, M = \ker \pi$, since $\ker \pi$ is maximal.)

An interesting question is whether a maximal superfluous cover (if it exists) of a module N need be unique up to isomorphism. A stronger requirement is that any superfluous cover must be "part" of a given maximal superfluous cover $\pi : M \to N$, in the sense that if $\pi' : M' \to N$ is any superfluous cover then there is an onto map $\varphi : M \to M'$ such that $\pi = \pi'\varphi$. Let us call this condition *strong uniqueness*. It is well-known that projective covers are, in fact, strongly unique. We next consider uniqueness and strong uniqueness for maximal covers of cyclic modules.

REMARK 1.12: Suppose $N = R/L$ cyclic, and $\pi : M \to N$ is a maximal superfluous cover. Write $M = R/L_1$. Strong uniqueness is equivalent to the condition:

If $L_2 \subseteq L$ such that $\sqrt{L_1} = \sqrt{L} = \sqrt{L_2}$ then $L_1 \subseteq L_2$.

Clearly this condition holds if $\sqrt{L_1 \cap L_2} = \sqrt{L}$. Although we do not see in general why this need hold, there is one special situation where it is true.

REMARK 1.13: Suppose $K \triangleleft R$, and R/K is a maximal superfluous cover of R/L. Then it is strongly unique. Indeed, as in Remark 1.8, $K = KL \triangleleft R$, and $KL \subseteq K \cap L$, so we are done by Remark 1.12.

Projectivity plays an additional role in the theory of superfluous covers. Recall that a module M is N-projective if for any epic $g : N \to N'$, each map $h : M \to N'$ lifts to a map $f : M \to N$ (i.e., $h = gf$).

PROPOSITION 1.14. *If M is N-projective then the following condition holds: For any superfluous cover $\pi : M' \to M$ and every $f : M' \to N$ we have $f(\ker \pi) = 0$.*

PROOF: Consider the diagram

$$\begin{array}{ccc} M' & \xrightarrow{f} & N \\ \pi\big\downarrow & & \big\downarrow\rho \\ M & \xrightarrow{\bar{f}} & N/f(\ker \pi) \end{array}$$

where $\rho : N \to N/f(\ker \pi)$ is the canonical map and $\bar{f}$ is given by viewing M as $M'/\ker \pi$ and defining $\bar{f}(x + \ker \pi) = \bar{f}(x) + \ker \pi$. Then $\bar{f}$ is onto, and by definition $\bar{f}$ lifts to a map $g : M \to N$ such that $\rho g = \bar{f}$. We would like $g\pi$ to equal f, since then we would be done. To this end, consider $f' = f - g\pi$. If $x \in M'$ then $\rho f'(x) = \rho f(x) - \bar{f}\pi(x) = 0$, implying

$f'(x) \in \ker\rho = f(\ker\pi)$. Write $f'(x) = f(k)$ where $k \in \ker\pi$. Then

$$f'(x-k) = f'(x) - f'(k) = f'(x) - f(k) = 0$$

so $x - k \in \ker f'$, proving $x \in \ker\pi + \ker f'$, i.e., $M' = \ker\pi + \ker f'$. But $\ker\pi$ is small in M' by hypothesis, so we conclude $\ker f' = M'$, i.e., $f = g\pi$ as desired. ∎

REMARK 1.16: The condition of Proposition 1.15 holds iff, for any superfluous cover $\pi: M' \to M$, each map $f: M' \to N$ drops down to a map $\overline{f}: M \to N$ such that $\overline{f}\pi = f$. (Indeed, $\overline{f}$ can be defined via Noether's homomorphism theorems iff $\ker\pi \subseteq \ker f$, i.e., iff $f(\ker\pi) = 0$.)

§2. Auto-covering cyclic modules.

Returning to example 1.7, Proposition 1.9, we are led to generalize [**8**, Theorem 2.1] in an attempt to characterize auto-covering cyclic modules, also cf. Remark 1.7.

Now let us define the following properties for a left ideal L.

P1. For every simple module S, every module map $\varphi: L \to S$ extends to a map $\psi: R \to S$, i.e., $\varphi = \psi|_L$.

P2. If $K \le L$ and $\sqrt{K} = \sqrt{L}$ then $K = L$.

THEOREM 2.1. *The following are equivalent for a left ideal* L :

(1) *If* $L' \le L$ *and* $\sqrt{L'} = \sqrt{L}$ *then* L' *satisfies P1 (in particular,* L *satisfies P1).*

(2) L *satisfies P2.*

(3) *The cyclic module* R/L *is auto-covering.*

PROOF: (1) ⇒ (2). Suppose on the contrary that $K < L$ with $\sqrt{K} = \sqrt{L}$. Fixing any a in $L \setminus K$, take $K_a > K$ maximal in L with respect to $a \notin K_a$. Let $L' = K_a + Ra$. Clearly, L'/K_a is simple, and $\sqrt{L'} = \sqrt{L}$ since $\sqrt{K} = \sqrt{L}$. Thus, by P1, the natural map $L' \to L'/K_a$ extends to a map $R \to L'/K_a$, which implies that the identity map on L'/K_a lifts to a map $R/K_a \to L'/K_a$, i.e., L'/K_a is a direct summand of R/K_a. Letting K'/K_a denote its complement, we see K' is a maximal left ideal of R containing K_a, but $K' \not\supseteq L'$, contrary to $\sqrt{K} = \sqrt{K_a} = \sqrt{L'} = \sqrt{L}$.

(2) ⇔ (3). By Example 1.6.

(2) $\Rightarrow$ (1). As in [8, Theorem 2.1]. First note by (2) that $L' = L$. Given $\varphi\colon L \to S$, note that $\ker\varphi$ is a left ideal of R properly contained in L, so by (2), there is a maximal left ideal M containing $\ker\varphi$ which does not contain L. Then $M + L = R$ and so $R/M \approx L/M \cap L$, a nonzero homomorphic image of $L/\ker\varphi \approx S$, so $R/M \approx S$ canonically, and the induced map $R \to S$ extends φ. □

NOTE: Condition (1) is somewhat artificial, since in fact the only L' which exists at the end is L itself. Thus, we would like to replace (1) by

(1′) L satisfies P1.

COROLLARY 2.2. *If L is f.g. then the conditions of Theorem 2.2 are all equivalent to* (1′).

PROOF: Obviously (1) $\Rightarrow$ (1′). Conversely, if L is f.g. then any submodule of L is contained in a maximal submodule of L, so using this submodule instead of K_a (and L itself instead of L') enables us to prove (1′) $\Rightarrow$ (2), exactly as in the proof of (1) $\Rightarrow$ (2) of Theorem 2.1. □

If we are willing to view all cyclic preimages of R/L at once, our results become somewhat sharper.

COROLLARY 2.3. *The following are equivalent, for an arbitrary left ideal L of R:*

(1) *Every left ideal $L' \leq L$ satisfies P1;*

(2) *Every left ideal $L' \leq L$ satisfies P2;*

(3) *R/L' is auto-covering for every $L' \leq L$;*

(4) *$L' = \sqrt{L'}$ for every $L' \leq L$.*

If any of these conditions hold then:

(5) *Every left ideal $L' \leq L$ is idempotent.*

PROOF: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) by Theorem 2.1.

(3) $\Leftrightarrow$ (4) by Example 1.6.

(3) $\Rightarrow$ (5) by Remark 1.7. □

REMARK 2.4: We examine (5) a bit closer, as in [8, lemma 2.3(c)]. Suppose the principal left ideal $Ra \leq L$ is idempotent. Then $a \in RaRa$, so $a = ba$ for some $b \in RaR$. In particular, if $a \in L \triangleleft R$ then $b \in L$. By [5,Exercise 13, p. 271], this proves that if $L \triangleleft R$ and Ra is idempotent for all a in L then R/L is flat as right R-module.

Accordingly, we say $L < R$ is talf if for all a in L there is b in L such that $a = ba$. (Thus, $L \triangleleft R$ is talf iff R/L is flat as a right module.)

PROPOSITION 2.5. *If $L < R$ is talf and $K \triangleleft R$ is contained in L, such that R/K is a superfluous cover of R/L, then $K = L$.*

PROOF: Passing to R/K, we may assume that $K = 0$. Thus, $L \subset \text{Jac}(R/K)$. Hence for any $a \in L$ we have $a = ba$ for some b in $\text{Jac}(R/K)$, so $(1-b)^{-1}$ exists in R/K and $a = (1-b)^{-1}(a - ba) = 0$.

In particular, if R is commutative then every flat cyclic module is auto-covering. This fails for non-commutative rings. For example, a regular ring which is not a V-ring will have (flat) cyclic modules which are not autocovering, in light of Proposition 1.8. ∎

An interesting question is whether every flat module over a commutative ring needs to be auto-covering. One can reduce this question rather easily to f.g. modules. Also, one may ask if all cyclic flat modules over a PI ring must be autocovering (By [2], every von Neumann regular PI-ring is a V-ring, so the counterexamples in §1 are not applicable). Another application of this line of reasoning is

PROPOSITION 2.6. *Suppose R is commutative. Then conditions (1)-(5) of Theorem 2.3 are equivalent, and each is equivalent to*

(6) *For every ideal $I \triangleleft R$ contained in L, the module R/I is flat.*

PROOF: We have the cycle of implications proved in Theorem 2.3.

(6) $\Rightarrow$ (3) by Proposition 2.5, taking $I = L$.

(5) $\Rightarrow$ (6) by Remark 2.4. □

Let us return to the noncommutative case.

PROPOSITION 2.7. *If R/K is a superfluous cover of R/L and L is talf then $a \in Ka$ for all a in L.*

PROOF: Write $a = ba$ for b in L. Then $b \in \sqrt{L} = \sqrt{K}$, so, by Example 1.4, there is c such that $c(1-b) - 1 \in K$. But $-(c(1-b)-1)a = -c(a - ba) - a = a$, so $a \in Ka$. ∎

One would like to conclude that $a \in K$; one situation in which this must be the case is when $a \in Z(R)$.

COROLLARY 2.8. *If $I \triangleleft R$ is generated by central elements and R/I is flat then R/I is auto-covering, $I \triangleleft R$, and*

§3. Maximal coverings.

Having established criteria for modules to be auto-covering, we still would like to know when a module can have a maximal cover, i.e., a cover which is auto-covering. Although we saw that maximal covers do not necessarily exist (e.g. for a cyclic module over a PID), there are some strong positive results. Of course, V-rings have maximal covers since every module is auto covering. On the other hand every projective cover is a maximal cover, so modules over perfect rings (and f.g. modules over semiperfect rings) have maximal covers. The main object of this section is to show that over many semilocal rings, the maximal covers of finitely generated modules are precisely the projective covers. Let us start with some general facts.

LEMMA 3.1. *If N_1, N_2 are auto-covering then $N_1 \oplus N_2$ is auto-covering.*

PROOF: Suppose on the contrary that $\pi\colon N \to N_1 \oplus N_2$ is a superfluous cover. Let $M_1 = \pi^{-1}(N_1)$, and π_1 be the restriction of π to M_1. We want to show that $\pi_1 : M_1 \to N_1$ is a superfluous cover since then, by hypothesis, $0 = \ker \pi_1$ and therefore, noting that $\ker \pi \subseteq M_1$, $\ker \pi = 0$.

Thus we need to show that if $K + \ker \pi_1 = M_1$ then $K = M_1$. Let σ_2 be the projection $N_1 \oplus N_2 \to N_2$, and consider the composite map

$$\varphi\colon N \xrightarrow{\pi} N_1 \oplus N_2 \xrightarrow{\sigma_2} N_2.$$

Clearly $K \subseteq M_1 = \ker \varphi$, so we have an induced onto map $\bar{\varphi}\colon N/K \to N_2$ with $\ker \bar{\varphi} = \frac{\ker \varphi}{K}$. We claim that $\ker \bar{\varphi}$ is small in N/K. Indeed if $K \leq L \leq N$ such that $L/K + \ker \bar{\varphi} = N/K$ then

$$N = L + \ker \varphi = L + M_1 = L + K + \ker \pi_1 = L + K + \ker \pi = L + \ker \pi,$$

so $N = L$ and thus $N/K = L/K$, as claimed.

Hence $\bar{\varphi}$ is a superfluous cover of N_2, so by hypothesis, $\ker \bar{\varphi} = 0$, i,e, $M_1 = K$, as desired. ∎

The following criterion will be used repeatedly.

LEMMA 3.2. *Suppose $\pi_i: N_i \to M_i$ are superfluous covers. Then*

$$\pi = \bigoplus_{i=1}^{t} \pi_i : \bigoplus_{i=1}^{t} N_i \to \bigoplus_{i=1}^{t} M_i$$

is a superfluous cover, which is maximal iff each π_i is maximal .

PROOF: Clearly, $\ker \pi = \oplus \ker \pi_i$ is a small submodule of $\oplus N_i$. The converse follows from Lemma 3.1 and induction on t. ∎

LEMMA 3.3. *Suppose R is semilocal and $K, L \triangleleft R$. Then the natural map $R/K \to R/L$ is a superfluous cover iff $K + \mathrm{Jac}(R) = L + \mathrm{Jac}(R)$.*

PROOF: By Remark 1.6, we need to show $\sqrt{K} = \sqrt{L}$ iff $K + \mathrm{Jac}(R) = L + \mathrm{Jac}(R)$. Since $K + \mathrm{Jac}(R) \subseteq \sqrt{K}$, it suffices to prove $K + \mathrm{Jac}(R) = \sqrt{K}$.

Assume first R is semilocal Passing to $\overline{R} = R/\mathrm{Jac}(R)$, it suffices to prove $\overline{K} = \sqrt{\overline{K}}$, but this is clear since $\overline{R}$ is semisimple Artinian.

On the other hand, if $K, L \triangleleft R$ then $\mathrm{Jac}(R/K) = (K + \mathrm{Jac}(R))/K$, so again $\sqrt{K} = K + \mathrm{Jac}(R)$. □

We say R is <u>weakly (left) semiperfect</u> if every cyclic R-module has a maximal superfluous cover; <u>weakly right perfect</u> is defined analogously for right modules.

THEOREM 3.4. *A semilocal ring R is semiperfect iff it is weakly left and right semiperfect.*

PROOF: ($\Rightarrow$) Clear, since left and right semiperfect are equivalent.

($\Leftarrow$) We want to prove that for any idempotent $\overline{a}$ in $\overline{R} = R/J$ there is an idempotent e of R with $\overline{e} = \overline{a}$. First take $b \in Ra$ such that $\overline{Rb} = \overline{Ra}$ with Rb minimal such, where $^-$ denotes the image in R/J; writing $\overline{a} = \overline{r_1 b}$ and replacing b by $r_1 b$ we may assume $\overline{a} = \overline{b}$. In particular, $Rb = Rb^2$ by minimality of Rb.

Next take $c \in bR$ such that $\overline{cR} = \overline{bR}$ with cR minimal such. As before, we may assume $\overline{c} = \overline{b} = \overline{a}$. Then $cR = c^2R$, so $\ell(c) \cap Rc = 0$ where $\ell(\ )$ denotes the left annihilator (since if $0 = (rc)c$ then $0 = rc^2R = rcR$, implying $rc = 0$).

We claim Rc is also auto-covering. Indeed, if $Rd \leq Rc$ with $\overline{Rd} = \overline{Rc} = \overline{Rb}$ then writing $c = br_2 \in Rbr_2 = Rb^2r_2 = Rbc$, we can write $d = b'c$ for b' in Rb, yielding $\overline{Rb'} = \overline{Rb'b} = \overline{Rb'c} = \overline{Rd} = \overline{Rb}$; thus $Rb' = Rb$ and $Rd = Rbc = Rbr_2 = Rc$, as desired.

In particular, $Rc = Rc^2$, so $c = rc^2$ for some r. Hence $c^2 = cc = crc^2$, so

$$c - crc \in \ell(c) \cap Rc = 0.$$

Thus $c = crc$, so rc is idempotent. But $\overline{rc}\ \bar{c} = \overline{rc^2} = \bar{c}$ and $\bar{c}\ \overline{rc} = \overline{crc} = \bar{c}$, so the idempotent $\overline{rc}$ must equal the idempotent $\bar{c} = \bar{a}$, as desired. □

This theorem raises the question of whether weakly left semiperfect implies weakly right semiperfect. This might be provable directly, along the lines of [**4**], but meanwhile we have

LEMMA 3.5. *A cyclic module over a semilocal ring R has a maximal superfluous cover iff it has a projective cover (in which case they are the same), provided R satisfies one of the following properties:*

(i) *Any chain $\ell(b) \subseteq \ell(b^2) \subseteq \ell(b^3) \subseteq \dots$ terminates, where $\ell(\ )$ denotes the left annihilator ideal;*

(ii) *$R/\operatorname{Nil}(R)$ satisfies (i).*

PROOF: The nontrivial direction is $(\Rightarrow)$.

(i) Given a module R/Ra we want to find a projective cover, i.e., R/Re where e is an idempotent of R for which $\sqrt{Re} = \sqrt{Ra}$, which by Lemma 3.3 means $Re + J = Ra + J$. Take $b \in Ra$ such that $\overline{Rb} = \overline{Ra}$ with Rb minimal such. As in the previous proof, we may assume $\bar{a} = \bar{b}$. By assumption $\ell(b^k) = \ell(b^{k+1})$ for some k; replacing b by b^k, we may assume $\ell(b) = \ell(b^2)$. Hence $Rb \cap \ell(b) = 0$, so we conclude the proof as we dud in Theorem 3.4, taking b instead of c.

(ii) We show R/N is idempotent-lifting over R/J, where $N = \operatorname{Nil}(R)$. Indeed, for any a in R such that $\bar{a} = a + J$ is idempotent, we take a maximal superfluous cover R/Rb of R/Ra and note that

$$\sqrt{(Rb+N)/N} = \sqrt{(Ra+N)/N}.$$

(Indeed $\subseteq$ is clear, and on the hand, $(\sqrt{Rb}) + N \subseteq \sqrt{Rb+N}$. Hence

$$\sqrt{(Ra+N)/N} = \sqrt{(\sqrt{Ra}+N)/N} = \sqrt{(\sqrt{Rb}+N)/N}$$
$$\subseteq \sqrt{(\sqrt{Rb+N})/N} = \sqrt{Rb+N}/N$$

as desired.) Now, the proof of (i) shows $Rb + N$ contains an idempotent $e + N$ lying over a, i.e., $e^2 - e \in N = \mathrm{Nil}(R)$, and $\bar{e} = \bar{a}$. Then $0 = (e^2 - e)^t = e^t - ef(e)$ for some polynomial f, the sum of whose coefficients are 1, so as in [**9**, Proposition 1.1.25] we have the idempotent $(ef(e))^t$ lifting $\bar{a}$. □

THEOREM 3.6. *Suppose R is a semilocal ring satisying condition* (i) *or* (ii) *of Lemma* 3.5. *Then the following are equivalent:*

(i) *every simple R-module has a maximal superfluous cover;*

(ii) *every cyclic R-module has a maximal superfluous cover;*

(iii) *every finitely generated R-module has a maximal superfluous cover;*

(iv) *R is semiperfect.*

PROOF: Clearly (iv) → (iii) → (ii) → (i). Finally, (i) implies that every simple R-module has a projective cover, by Lemma 3.5, and thus that R is semiperfect. □

§4. Superfluous Covers of modules over Dedekind Domains.

Throughout this section, R is a Dedekind domain. Our first goal is to show that all torsion-free R-modules are autocovering.

LEMMA 4.1. *Superfluous covers of injective R-modules are injective.*

PROOF: Over a Dedekind domain, a module is injective if and only if it is divisible. Let $K \ll B$ and B/K be divisible. Let $r \in R$, $r \neq 0$. Since $r(B/K) = B/K$, it follows that $rB + K = B$. But then $rB = B$ and B is injective, as claimed.

LEMMA 4.2. *Every injective torsion-free R-module is autocovering.*

PROOF: Let A be an injective torsion-free R-module and let B be a superfluous cover of A, with kernel $K \ll B$. Then $E(K) \oplus L = B$, for some $L \subset B$. Therefore

$$\frac{E(K) \oplus L}{K} = \frac{B}{K} \cong A.$$

But then $A \cong E(K)/K \oplus L$. Since $E(K)/K$ is torsion, we conclude that $E(K)/K = 0$. Therefore, $K = 0$.

We prove next that over a Dedekind domain the classes of autocovering and flat modules coincide. This was also shown to be the case over left perfect rings by Bass (see [**1**, Theorem 28.4]).

THEOREM 4.3. *An R-module is autocovering if and only if it is torsion free.*

PROOF: Let M be a torsion-free R-module and let B be a superfluous cover for M with kernel $K \ll B$. Then $K \ll E(B)$. Since B/K is contained in the injective module $\frac{E(B)}{K}$, there exists $B \subset S \subset E(B)$ such that $E(B/K) = \frac{S}{K}$. But then $K \ll S$. So, S is a superfluous cover for the injective torsion-free module $E(B/K) \simeq E(M)$. By Lemma 4.2, $K = 0$.

To prove the converse, assume $0 \neq t(M)$ is the torsion submodule of a module M.

Arguing as in [**6**, Thm 9], one gets that $t(M)$ contains a summand of M which is either of the form R/P^n ($n \geq 1$, $0 \neq P$ a prime ideal of R) or R) or $E(R/P)$ ($0 \neq P$ a prime ideal of R).

Since R/P^{n+1} is a non-trivial superfluous cover of R/P^n and $E(R/P) \simeq (R/P^2)$ is a non-trivial superfluous cover of $E(R/P)$, we see that M is not autocovering, proving our claim.

The previous theorem allows us to provide an example of a module M over the ring of integers having a maximal cover which is not projective.

EXAMPLE 4.4. *Let $M = \bigoplus_p \mathbf{Z}_{p^\infty}$, where the sum runs over all prime integers. Then $\mathbf{Q}$ is a maximal superfluous cover of M.*

PROOF: Since $\mathbf{Q}$ has no maximal $\mathbf{Z}$- submodules, Rad $\mathbf{Q} = \mathbf{Q}$. So, $\mathbf{Q}$ has small submodules. In particular, $\mathbf{Q}$ has a cyclic small submodule $a\mathbf{Z}$. The isomorphism between $a\mathbf{Z}$ and $\mathbf{Z}$ extends to an automorphism of $\mathbf{Q}$. So, $\mathbf{Z} \ll \mathbf{Q}$. Since $\mathbf{Q}/\mathbf{Z} \cong M$ and $\mathbf{Q}$ is autocovering, $\mathbf{Q}$ is a maximal cover of M.

Let us turn next to the study of injective torsion R-modules. One may be tempted to study modules M over which every superfluous cover S of M is isomorphic to M. We will refer to these modules as 'autocovering up to isomorphism '. Indeed, every autocovering module is autocovering up to isomorphism. Our next Lemma shows that the converse is not true.

LEMMA 4.5. *Every indecomposable injective torsion R-module is autocovering up to isomorphism.*

PROOF: Let A be an indecomposable injective torsion R- module. Suppose B is a superfluous cover of A with kernel $K \ll B$. Write $B = \bigoplus_{i \in I} K_i$ as a sum of indecomposable injective modules. Consider the projections $\pi_i : B \to K_i$. Then $\pi_i(K) \ll K_i$, for all $i \in I$. Also, $K \subset \bigoplus_{i \in I} \pi_i(K)$ and, therefore, there is an epimorphism from $A \cong B/K$ onto

$$\frac{B}{\bigoplus \pi_i(K)} = \frac{\bigoplus K_i}{\bigoplus \pi_i(K)} \cong \bigoplus \frac{K_i}{\pi_i(K)} \cong \bigoplus K_i.$$

Since A is uniserial, $|I| = 1$ and therefore $B \cong A$.

The following is an easy proof of a well-known result, given here for the convenience of the reader.

LEMMA 4.6. *For $0 \neq K \ll \mathbf{Q}$, $\mathbf{Q}/K \cong \mathbf{Q}/\mathbf{Z}$.*

PROOF: Let $0 \neq a\mathbf{Z} \subset K \subseteq \mathbf{Q}$. Then $Q/K \cong \frac{\mathbf{Q}/a\mathbf{Z}}{K/a\mathbf{Z}}$ and $K/a\mathbf{Z} \ll \mathbf{Q}/a\mathbf{Z} \cong \mathbf{Q}/\mathbf{Z}$. So, it suffices to show that if S is a small $\mathbf{Z}$- submodule of $T = \bigoplus_p \mathbf{Z}_{p^\infty}$ ($\cong \mathbf{Q}/\mathbf{Z}$), then $T/S \cong T$. First of all, since T/S is a homomorphic image of T, every $\mathbf{Z}_{p^\infty}$ appears as a summand of T/S at most once. On the other hand, let $\pi_p : T \to \mathbf{Z}_{p^\infty}$ be the natural projection map. Then $\pi_p(S) \ll \mathbf{Z}_{p^\infty}$. Also, $S \subset \bigoplus_p \pi_p(S)$ and, therefore, T/S maps onto $\frac{T}{\bigoplus \pi_p(S)} \cong \bigoplus \frac{\mathbf{Z}_{p^\infty}}{\pi_p(S)} \cong \bigoplus_p \mathbf{Z}_{p^\infty} = T$. Consequently, each $\mathbf{Z}_{p^\infty}$ appears at least once as a summand of T/S. Therefore $T/S \cong T$, as claimed.

We show next that the direct sum of two autocovering up to isomorphism modules need not be autocovering up to isomorphism.

REMARK 4.7. *There exist two $\mathbf{Z}$-modules M and N which are autocovering up to isomorphism but $M \oplus N$ is not autocovering up to isomorphism.*

PROOF: Let $M = \mathbf{Z}_{2^\infty}$) and $N = \bigoplus_{p \neq 2} \mathbf{Z}_{p^\infty}$. M is autocovering up to isomorphism, by Lemma 4.5. Let S be a superfluous cover for N with kernel $K \ll S$. Then $S = \bigoplus_{i \in I} E_i$, a sum of indecomposable injective modules. We claim each E_I is torsion. Otherwise, let $\mathbf{Q} \subset^{\oplus} S$. Then there is a nonzero map $\varphi : \mathbf{Q} \to N$ with small kernel $L \ll \mathbf{Q}$. But then $\mathbf{Q}/L$ embeds in N and $\mathbf{Q}/L \cong \mathbf{Q}/\mathbf{Z}$ (by Lemma 4.6). This is a contradiction since the

2-primary component of N is zero. So, as claimed, every E_i is torsion. Clearly, $\mathbf{Z}_{2^\infty}$ is not a summand of S, while, for every $p \neq 2$, $\mathbf{Z}_{p^\infty}$ must be a summand of S. We need only to rule out the possibility that $\mathbf{Z}_{p^\infty} \times \mathbf{Z}_{p^\infty}$ may be a summand of S. This is not the case since otherwise there is a map $\Psi : \mathbf{Z}_{p^\infty} \times \mathbf{Z}_{p^\infty} \to N$ with small kernel K'. But then $\frac{\mathbf{Z}_{p^\infty} \times \mathbf{Z}_{p^\infty}}{K'}$ embeds in N. This is a contradiction since $\frac{\mathbf{Z}_{p^\infty} \times \mathbf{Z}_{p^\infty}}{K'} \cong \mathbf{Z}_{p^\infty} \times \mathbf{Z}_{p^\infty}$. So, $S \cong N$ and N is autocovering up to isomorphism.

Acknowledgements:. The authors would like to thank Professor K.M. Rangaswami for helpful conversations on §4. This paper originated during the authors' visit to the Centre de Recerca Matemàtica in Bellaterra (Barcelona) during the Summer of 1992. We would like to thank the staff of the CRM for their kind hospitality.

We also would like to thank the referee for his helpful remarks.

References

1. Anderson, F. and Fuller, K., "Rings and Categories of Modules," Springer-Verlag, Berlin, 1974.
2. Armendariz, E., Fisher, J.W., *Regular PI rings*, Proc. Amer. Math. Soc. **39** (1973), 247-251.
3. Cozzens, J.H., *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc. **76** (1970), 75-79.
4. Dischinger, F., *Sur les anneaux fortement π-réguliers*, C.R. Acad. Sci. Paris Sér. A-B **283** (1976), A571-A573.
5. Fisher, J.W., *Von Neumann Regular Rings vs V rings*, Dekker Lecture Notes **7**, 101-119.
6. Kaplansky, I., *Infinite Abelian groups*, University of Michigan Pren, Ann Arbor, (1969).
7. Kasch, "Modules and Rings," Academic Press, Inc., London, New York, 1982.
8. Michler G.O. and Villamayor O.E., *On rings whose simple modules are injective*, J. Algebra **25** (1973), 185 201.
9. Rowen L.H., "Ring Theory, vol.1," Pure and Applied Math. **127**, Academic Press, Boston, 1988.

Received: March 1994

Revised: October 1994 and December 1994