

## WEAKLY-INJECTIVE MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS

S. K. JAIN and S. R. LÓPEZ-PERMOUTH

(Received 1 February 1992; revised 21 July 1992)

Communicated by P. Schultz

### Abstract

A module  $M$  is said to be weakly-injective if and only if for every finitely generated submodule  $N$  of the injective hull  $E(M)$  of  $M$  there exists a submodule  $X$  of  $E(M)$ , isomorphic to  $M$  such that  $N \subset X$ . In this paper we investigate weakly-injective modules over bounded hereditary noetherian prime rings. In particular we show that torsion-free modules over bounded hnp rings are always weakly-injective, while torsion modules with finite Goldie dimension are weakly-injective only if they are injective.

As an application, we show that weakly-injective modules over bounded Dedekind prime rings have a decomposition as a direct sum of an injective module  $B$ , and a module  $C$  satisfying that if a simple module  $S$  is embeddable in  $C$  then the (external) direct sum of all proper submodules of the injective hull of  $S$  is also embeddable in  $C$ . Indeed, we show that over a bounded hereditary noetherian prime ring every uniform module has periodicity one if and only if every weakly-injective module has such a decomposition.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary: 16A14, 16A52; secondary: 16A12, 16A33.

*Keywords and phrases*: bounded hereditary noetherian prime rings, weakly-injective modules.

### 1. Introduction

The study of hereditary noetherian prime (hnp) rings generalizes that of bounded Dedekind prime rings and in particular of their best known example, the ring of integers  $\mathbb{Z}$ . These rings and their modules have been studied extensively; see [3, 2, 4, 8], for example. McConnell and Robson's book [10] has a nice chapter on hnp and related rings. In [8], Lenagan proved that an hnp ring is either primitive or bounded. Special classes of modules over bounded hnp rings (including injective, projective, quasi-injective and quasi-projective) have been studied in [4, 9, 12, 13, 14]. In this paper

we discuss weakly-injective modules over bounded hnp rings.

Given an arbitrary ring  $R$  and  $R$ -modules  $M$  and  $N$ , we say that  $M$  is *weakly  $N$ -injective* if and only if every map  $\varphi : N \rightarrow E(M)$  from  $N$  into the injective hull  $E(M)$  of  $M$  may be written as a composition  $\sigma \circ \hat{\varphi}$  where  $\hat{\varphi} : N \rightarrow M$  is a homomorphism and  $\sigma : M \rightarrow E(M)$  is a monomorphism. This is equivalent to saying that for every map  $\varphi : N \rightarrow E(M)$  there exists a submodule  $X$  of  $E(M)$ , isomorphic to  $M$  such that  $\varphi(N)$  is contained in  $X$ . In particular,  $M$  is weakly  $R$ -injective if and only if for every  $x \in E(M)$  there exists  $X \subset E(M)$  such that  $x \in X \cong M$ . We say that  $M$  is *weakly-injective* if and only if it is weakly  $N$ -injective for every finitely generated module  $N$ . Clearly,  $M$  is weakly-injective if and only if for every finitely generated submodule  $N$  of  $E(M)$  there exists  $X \subset E(M)$  such that  $N \subset X \cong M$ .

Any weakly  $N$ -injective module  $M$  satisfies the closely related property that for every submodule  $K$  of  $N$ , if  $N/K$  embeds in  $E(M)$  then  $N/K$  embeds in  $M$ . Following [5], we refer to any such module as being  *$N$ -tight*. If  $M$  is  $N$ -tight for every finitely generated module  $N$ , we simply say that  $M$  is *tight*.

Weakly-injective (tight) modules are closed under finite sums and under essential extensions. However, they remarkably fail to be closed under direct summands [7]. Furthermore, arbitrary sums of weakly-injective right modules over a ring  $R$  are weakly-injective if and only if  $R$  is a right q.f.d. ring (that is, all cyclic right  $R$ -modules have finite Goldie dimension) [1].

Throughout all rings have 1 and all modules are right unital modules unless otherwise stated. If  $N$  is a submodule of  $M$ ,  $N \subset' M$  will mean that  $N$  is essential in  $M$ .

## 2. Preliminaries

The exact relation between weak relative-injectivity and relative tightness is given in the following lemma from [7].

LEMMA 2.1. *Given two modules  $M$  and  $N$ ,  $M$  is weakly  $N$ -injective if and only if for every submodule  $K \subset N$  and for every monomorphism  $\varphi : N/K \rightarrow E(M)$ :*

- (1) *there exists a monomorphism  $\varphi' : N/K \rightarrow M$ , and*
- (2) *for every complement  $L$  of  $\varphi'(N/K)$  in  $M$  there exists  $K' \subset E(M)$  such that  $K' \cap \varphi(N/K) = 0$  and  $K' \cong L$ .*

PROOF. See [7, Lemma 1.3].

It follows easily from the previous lemma that a uniform module  $U$  is weakly-injective if and only if it is tight. As a matter of fact, for any module  $M$ , if  $E(M)$  is a

direct sum of indecomposables,  $M$  is tight if and only if it is weakly injective. This is the subject of our next proposition.

**PROPOSITION 2.2.** *Let  $M$  be an  $R$ -module such that the injective hull  $E(M)$  of  $M$  is a direct sum of indecomposables. Then  $M$  is tight if and only if it is weakly-injective.*

**PROOF.** Let  $M$  be a tight right  $R$ -module such that  $E(M)$  equals a direct sum of indecomposables, say  $E(M) = \bigoplus_{i \in I} E_i$ . Let  $N$  be a finitely generated submodule of  $E(M)$ . Then there exists a finite subset  $J \subset I$  such that  $N \subset \bigoplus_{i \in J} E_i$ . Without loss of generality we may assume that  $E(N) = \bigoplus_{i \in J} E_i$ . Let  $\varphi : N \rightarrow M$  be an embedding of  $N$  into  $M$  as is guaranteed by the tightness of  $M$ . Then  $E(M) = E(\varphi(N)) \oplus K$ , for some submodule  $K \subset E(M)$ . It follows from the Azumaya-Krull-Schmidt theorem that  $K \cong \bigoplus_{i \in I-J} E_i$ . Let  $A = M \cap K$ . Then  $A \subset K$  and hence  $\varphi(N) \oplus A$  may be embedded in  $E(M)$  via a map  $\sigma$  such that  $N = \sigma(\varphi(N))$ . By the injectivity of  $E(M)$  and the essentiality of the inclusion  $\varphi(N) \oplus A \subset M$ , we obtain a monomorphism  $\hat{\sigma} : M \rightarrow E(M)$ , extending  $\sigma$ , such that  $N \subset \hat{\sigma}(M)$ , as desired.

Proposition 2.2 has the following immediate corollary.

**COROLLARY 2.3.** *For a right noetherian ring  $R$ , a right  $R$ -module is weakly-injective if and only if it is tight.*

**PROOF.** Obvious.

The following lemmas, due to Singh, are listed here without proof for easy reference.

**LEMMA 2.4.** *Let  $R$  be a bounded hnp ring and let  $E$  be an indecomposable injective torsion right  $R$ -module. Then  $E$  has a unique chain of submodules*

$$0 = x_0 R \subset x_1 R \subset x_2 R \subset \cdots \subset x_n R \subset \cdots$$

whose union is  $E$  such that

- (1) each  $x_{i+1} R / x_i R$  is a simple  $R$ -module;
- (2) the members of the chain are the only submodules of  $E$  different from  $E$ ; and
- (3) there exists a positive integer  $n$  such that for any  $i, j$ ,  $x_{i+1} R / x_i R \cong x_{j+1} R / x_j R$  if and only if  $i \equiv j \pmod{n}$ .

**PROOF.** See [12, Theorem 4] and [14, Corollary 2.9].

DEFINITION 2.5. Let  $E$  be an indecomposable injective torsion right  $R$ -module over a bounded hnp ring  $R$ . The unique infinite ascending chain of submodules of  $R$  described in Lemma 2.4 is called the *composition series* of  $E$  and the positive integer  $n$  is referred to as the *periodicity* of  $E$ . Furthermore, for any uniform module  $U$  over  $R$ , the periodicity of  $U$  is defined to be the periodicity of  $E(U)$ .

LEMMA 2.6. *For any uniform right  $R$ -module over a bounded Dedekind prime ring  $R$  the periodicity of  $U$  is 1.*

PROOF. See [12, Corollary 1].

DEFINITION 2.7. Let  $R$  be a bounded hnp ring. Two indecomposable injective torsion right  $R$ -modules are *equivalent* if they are homomorphic images of each other. Due to the finite periodicity, this is indeed equivalent to requiring that one of them be a homomorphic image of the other. Two torsion uniform modules are equivalent if their injective hulls are equivalent. Furthermore, two uniform elements  $x$  and  $y$  in a torsion right  $R$ -module are said to be equivalent if  $xR$  and  $yR$  are equivalent uniform right  $R$ -modules. A torsion right  $R$ -module  $M$  is said to be *primary* if every pair of uniform elements of  $M$  is equivalent. Given a uniform element  $x$  in a torsion  $R$ -module  $M$ , the submodule  $N$  of  $M$  generated by all the uniform elements of  $M$  equivalent to  $x$  is primary. Such an  $N$  is called a (the) *primary component* of  $M$  (corresponding to  $x$ ).

LEMMA 2.8. *Every torsion module over a bounded hnp ring is the direct sum of its primary components.*

PROOF. See [13, Lemma 9].

We believe that the following result must be well-known but we have not been able to find it anywhere in the literature. We include it here without a proof.

LEMMA 2.9. *Let  $A$  be a submodule of a module  $B$ , and let  $n \in \mathbb{Z}^+$ . Then  $\text{Soc}^n A = A \cap \text{Soc}^n B$  and  $\text{Soc}^n A / \text{Soc}^{n-1} A$  is embeddable in  $\text{Soc}^n B / \text{Soc}^{n-1} B$ .*

### 3. Weakly-injective modules over bounded HNP rings

It has been shown that any noetherian prime ring is a weakly-injective ring (i.e. it is weakly-injective as a module over itself) [7]. Indeed, more is true:

PROPOSITION 3.1. *Every torsion-free module over a noetherian prime ring is weakly-injective.*

PROOF. Over a noetherian prime ring  $R$  every torsion-free right module contains an essential submodule which is a direct sum of uniform submodules. Since weakly-injective modules over noetherian rings are closed under arbitrary direct sums and under essential extensions, it suffices to show that every uniform right  $R$ -module is weakly-injective. Let  $U$  be a uniform right  $R$ -module and let  $V$  be a finitely generated submodule of  $E(U)$ . Since  $R$  is prime and noetherian it follows that  $V$  is isomorphic to a right ideal of  $R$  and that therefore it embeds in  $U$ . In light of Corollary 2.3, this completes our proof.

The above proposition has the following corollary.

**COROLLARY 3.2.** *For any module  $A$  over a noetherian prime ring  $R$ ,  $A$  is weakly-injective if and only if its singular submodule  $Z(A)$  is weakly-injective.*

PROOF. The injective hull of  $A$  may be written as  $E(A) = E(Z(A)) \oplus K$ , where  $Z(A)$  is the torsion submodule of  $A$  and  $K$  is some submodule of  $E(A)$ . If  $A$  is weakly injective and  $N$  is a finitely generated submodule of  $E(Z(A))$  then  $N$  embeds in  $A$ . But  $N$  is itself torsion and hence  $N$  embeds in  $Z(A)$ . In light of Corollary 2.3 this proves our claim that  $Z(A)$  is weakly-injective. On the other hand, if  $Z(A)$  is weakly-injective then  $A$  must be also weakly-injective since it contains as an essential submodule the direct sum of weakly-injective modules  $Z(A) \oplus (K \cap A)$ .

Due to the above corollary, in order to characterize weakly-injective modules over bounded hnp rings it suffices to center our attention on torsion modules.

By Lemma 2.8, any torsion module over a bounded hnp ring can be expressed as the direct sum of its primary components. While weak-injectivity does not usually come down to summands, we have the following result.

**LEMMA 3.3.** *A torsion module over a bounded hnp ring is weakly-injective if and only if its primary components are weakly-injective.*

PROOF. Let  $A$  be a torsion module over the bounded hnp ring  $R$ . By Lemma 2.8, we may write  $A = \bigoplus_{i \in I} A_i$ , where the  $A_i$ 's are the primary components of  $A$ . Since sums of weakly-injective modules over noetherian rings are weakly-injective we only need to show that if  $A$  is weakly-injective so is  $A_j$  for each  $j \in I$ . Let  $N$  be a finitely generated submodule of  $E(A_j) \subset E(A) = \bigoplus_{i \in I} E(A_i)$ . Clearly, for every  $i \in I$ ,  $E(A_i)$  is a primary component of  $E(A)$ . By the weak-injectivity of  $A$  there exists an embedding  $\varphi : N \rightarrow A$ . Since  $\varphi(N) \cong N \subset E(A_j)$  it follows that the uniform elements in  $\varphi(N)$  are equivalent to those in  $A_j$ . Hence  $\varphi(N) \subset A_j$ . So  $A_j$  is tight and therefore, due to Corollary 2.3, weakly-injective as claimed.

The above lemma has, as an immediate application, the following characterization of weakly-injective torsion modules with finite Goldie dimension.

LEMMA 3.4. *If a torsion module  $A$  over a bounded hnp ring has finite Goldie dimension, then  $A$  is weakly-injective only if it is injective.*

PROOF. Let  $R$  be a bounded hnp ring and let  $A$  be a torsion right  $R$ -module with finite Goldie dimension  $n$ . Assume that  $A$  is weakly-injective. Since  $\text{Soc}A \subset' A$ , we may write  $\text{Soc}A = S_1 \oplus \cdots \oplus S_n$ , where for every  $i = 1, \dots, n$ ,  $S_i$  is simple. For every  $i = 1, \dots, n$ , let  $0 \subset a_{i1}R \subset a_{i2}R \subset \cdots$  be the composition series of  $E(S_i)$ . Then for every  $m \in \mathbb{Z}^+$ ,  $\text{Soc}^m E(A) = a_{1m}R \oplus a_{2m}R \oplus \cdots \oplus a_{nm}R$ . It follows that  $E(A) = \bigcup_{m=1}^{\infty} \text{Soc}^m E(A)$ . So, in order to prove that  $A$  is injective it suffices to prove that for every  $m \in \mathbb{Z}^+$ ,  $\text{Soc}^m E(A) = \text{Soc}^m A$ . Since  $A$  is weakly-injective, for every  $n \in \mathbb{Z}^+$  there exists an embedding  $\varphi : \text{Soc}^n E(A) \rightarrow A$ . We will first prove by induction that for every embedding  $\varphi : \text{Soc}^m E(A) \rightarrow A$ ,  $\varphi(\text{Soc}^m E(A)) = \text{Soc}^m A = \text{Soc}^m E(A)$ . The result is clear if  $m = 1$ . Suppose it is true for  $m = j - 1$  and assume that  $\varphi : \text{Soc}^j E(A) \rightarrow A$  is an embedding. By the inductive hypothesis, the restriction of  $\varphi$  to  $\text{Soc}^{j-1} E(A)$  is an isomorphism onto  $\text{Soc}^{j-1} A = \text{Soc}^{j-1} E(A)$ . Then

$$(1) \quad \begin{aligned} \frac{\text{Soc}^j E(A)}{\text{Soc}^{j-1} E(A)} &\cong \frac{\varphi(\text{Soc}^j E(A))}{\varphi(\text{Soc}^{j-1} E(A))} = \frac{\varphi(\text{Soc}^j E(A))}{\text{Soc}^{j-1} E(A)} \\ &\subset \text{Soc} \left( \frac{A}{\text{Soc}^{j-1} E(A)} \right) \subset \text{Soc} \left( \frac{E(A)}{\text{Soc}^{j-1} E(A)} \right). \end{aligned}$$

From the first inequality in (1),  $\varphi(\text{Soc}^j E(A)) \subset \text{Soc}^j A$ . Also by (1), the Goldie dimension of  $\text{Soc}^j A / \text{Soc}^{j-1} A$  is at least  $n$ , since  $\text{Soc}^j E(A) / \text{Soc}^{j-1} E(A) = \sum_{i=1}^n a_{ij}R / a_{i,j-1}R$ , a direct sum of  $n$  simples. On the other hand, Lemma 2.9 implies that the Goldie dimension of  $\text{Soc}^j A / \text{Soc}^{j-1} A$  is at most equal to the Goldie dimension of  $\text{Soc}^j E(A) / \text{Soc}^{j-1} E(A)$ , which equals  $n$ . So, using (1) once again, we obtain  $\varphi(\text{Soc}^j E(A)) / \text{Soc}^{j-1} E(A) = \text{Soc}^j A / \text{Soc}^{j-1} A$  and hence  $\varphi(\text{Soc}^j E(A)) = \text{Soc}^j A = \text{Soc}^j E(A)$ , as desired. This concludes our induction.

Weakly-injective torsion modules with infinite Goldie dimension will be characterized in the next lemma but first we need to introduce some notation. Let  $S$  be a simple module over a bounded hnp ring  $R$ . We define  $N_S$  to be the serial module consisting of the external direct sum of all proper submodules of  $E(S)$ . Namely,

$$N_S = \bigoplus_{\substack{B \subsetneq E(S)}} B.$$

LEMMA 3.5. *Let  $A$  be a torsion module with homogeneous socle and infinite Goldie dimension. The following statements are equivalent:*

- (1)  $A$  is weakly-injective.
- (2) For any simple module  $S$ , if  $S$  embeds in  $A$  then  $N_S$  embeds in  $A$ .
- (3) For every  $n \in \mathbb{Z}^+$ ,  $\text{Soc}^n(A)/\text{Soc}^{n-1}(A)$  is infinite dimensional.

PROOF. Let  $S$  be a simple submodule of  $A$ . From the hypotheses, the injective hull of  $A$  is a direct sum of infinitely many copies of  $E(S)$ . By Lemma 2.4,  $E(S)$  has a composition series  $0 \subset S = x_1R \subset x_2R \subset \cdots \subset E(S)$ . Clearly, any finitely generated submodule of  $E(A)$  can be embedded in  $N_S$  and therefore (2) implies (1). If we assume that  $A$  is weakly-injective then for every  $m, n \in \mathbb{Z}^+$ , the finitely generated module  $(x_mR)^n$  is embeddable in  $A$ . In light of Lemma 2.9 this implies that for every  $m, n \in \mathbb{Z}^+$ , the Goldie dimension of  $\text{Soc}^m A/\text{Soc}^{m-1}A$  is larger than  $n$  and hence it must be infinite. Thus (1) implies (3). So it is only left to show that (3) implies (2). Let us assume that for every  $m \in \mathbb{Z}^+$ ,  $\text{Soc}^m A/\text{Soc}^{m-1}A$  is infinite dimensional. We shall proceed inductively to construct an ascending sequence of submodules of  $A$ ,  $0 = N_0 \subset N_1 \subset N_2 \subset \cdots$  such that, for every  $i \in \mathbb{Z}^+$ ,  $N_i = N_{i-1} \oplus y_iR$ , for some  $y_i \in A$  such that  $y_iR \cong x_iR$ . Obviously,  $N = \bigcup_{i=1}^{\infty} N_i$  will then be a submodule of  $A$  isomorphic to  $N_S$ , proving our claim. For  $n = 1$ , since  $\text{Soc}(A) \neq 0$  we have a simple submodule  $0 \neq y_1R$  of  $A$ . Since the socle is homogeneous,  $y_1R \cong x_1R$ . Thus, let  $N_1 = y_1R$ . Suppose  $N_{m-1}$  has been constructed, then  $N_{m-1} \cong x_1R \oplus x_2R \oplus \cdots \oplus x_{m-1}R$ . Since  $\text{Soc}^m A/\text{Soc}^{m-1}A$  is infinite dimensional, it has a submodule consisting of a sum of  $m$  simple submodules, say  $S_1 \oplus S_2 \oplus \cdots \oplus S_m \subset \text{Soc}^m A/\text{Soc}^{m-1}A$ . Let us write  $S_i = \bar{z}_iR$ , where  $\bar{z}_i = z_i + \text{Soc}^{m-1}A$  (for some  $z_i \in \text{Soc}^m A$ ). The finitely generated submodule  $z_1R + \cdots + z_mR$  of  $A$ , being torsion, is equal to a direct sum  $t_1R \oplus \cdots \oplus t_kR$  of cyclic submodules (See [12, Lemma 1], for example). One can easily check that (i)  $k \geq m$ , (ii) for each  $i = 1, \dots, k$  there exists  $1 \leq j \leq m$  such that  $t_iR \cong x_jR$ , and (iii) there exist exactly  $m$   $t_i$ 's such that  $t_iR \cong x_mR$ , say,  $t_{i_1}, t_{i_2}, \dots$  and  $t_{i_m}$ . Among  $t_{i_1}R, t_{i_2}R, \dots$  and  $t_{i_m}R$  there exists at least one whose intersection with  $N_{m-1}$  is zero (otherwise the socle of  $N_{m-1}$  would contain a direct sum of  $m$  distinct simple submodules). Let  $t_{i_j}R$  be one such module, then let  $y_m = t_{i_j}$  and define  $N_m = N_{m-1} \oplus y_mR$ . This completes the proof of our lemma.

#### 4. Bounded HNP rings whose uniform modules have periodicity one

THEOREM 4.1. *Let  $A$  be a right module over a bounded hnp ring. If all uniform submodules of  $A$  have periodicity one, then the following statements are equivalent:*

- (1)  $A$  is weakly-injective.

- (2) *There is a decomposition  $A = B \oplus C$  such that (i)  $B$  is torsion, injective and has finite dimensional primary components, (ii)  $C$  satisfies that if a simple module  $S$  embeds in  $C$  then the module  $N_S$  embeds in  $C$ , and (iii)  $B$  and  $C$  have no isomorphic simple submodules.*
- (3) *There is a decomposition  $A = B \oplus C$  such that  $B$  is injective and  $C$  satisfies that if a simple module  $S$  embeds in  $C$  then the module  $N_S$  embeds in  $C$ .*

PROOF. Let  $A$  be a right module over a bounded hnp ring  $R$ . If  $A$  is weakly-injective, so is  $Z(A)$  (Corollary 3.2), and also so are the primary components of  $Z(A)$  (Lemma 3.3). Let  $B$  be the (direct) sum of all the primary components of  $Z(A)$  with finite Goldie dimension. By Lemma 3.4, each such primary component is injective and therefore so is  $B$ . It follows that we may write  $A = B \oplus C$ , where  $C$  is chosen so that it contains the primary components of  $Z(A)$  not already contained in  $B$ . If  $S$  is a simple module and a monomorphism  $\varphi$  embeds  $S$  in  $C$  then  $S$  actually embeds in the primary component  $N$  (say) of  $Z(A)$  corresponding to  $\varphi(S)$ . By the weak-injectivity of  $N$  and in light of Lemma 3.5, we conclude that  $N_S$  embeds in  $N$  and consequently in  $C$ , as claimed. The decomposition  $A = B \oplus C$  satisfies conditions (i), (ii) and (iii) in (2) and therefore we conclude that (1) implies (2). Obviously (2) implies (3). The conditions in (3) imply that  $Z(C)$  is weakly-injective (by Lemma 3.5). Therefore, by Corollary 3.2,  $C$  is weakly-injective and hence  $A$ , being the sum of two weakly-injective modules, is weakly-injective. Thus, (3) implies (1).

COROLLARY 4.2. *The statements in Theorem 4.1 about a right module  $A$  over the ring  $R$  are equivalent if  $R$  is a bounded Dedekind prime ring.*

PROOF. Lemma 2.6 guarantees that if  $R$  is a bounded Dedekind prime ring, then  $A$  satisfies the hypotheses of the theorem.

Let  $R$  be a bounded hnp ring and let  $E$  be an indecomposable injective right  $R$ -module with periodicity  $\geq 2$ . Let  $0 \subset x_1R \subset x_2R \subset \cdots \subset E$  be the composition series of  $E$ . Then  $x_1R \not\cong x_2R/x_1R$ . We refer to  $E(x_2R/x_1R) = E/x_1R$  as  $\bar{E}$  and, for each  $x \in E$ ,  $\bar{x}$  denotes  $x + x_1R \in \bar{E}$ . For every  $j \in \mathbb{Z}^+$ , let  $M_j$  be the submodule of  $E \oplus \bar{x}_2R \oplus \cdots \oplus \bar{x}_jR$  consisting of those elements  $(a_1, \bar{a}_2, \dots, \bar{x}_j)$  such that  $\bar{a}_1 = \bar{a}_2 + \cdots + \bar{a}_j$ . Also, let  $M$  be the submodule of the infinite sum  $E \oplus \bar{x}_2R \oplus \bar{x}_3R \oplus \cdots$  consisting of those elements  $(a_1, \bar{a}_2, \bar{a}_3, \dots)$  such that  $\bar{a}_1 = \sum_{i=2}^{\infty} \bar{a}_i$ . For convenience we shall employ the usual unit vectors (sequences),  $e_i = (0, 0, \dots, 0, 1, 0, \dots)$  where the only 1 is in the  $i$ -th place as a notational device so that we may write  $(a_1, \bar{a}_2, \dots, \bar{a}_j) = a_1e_1 + \bar{a}_2e_2 + \cdots + \bar{a}_je_j$  in  $M_j$  and also  $(a_1, \bar{a}_2, \bar{a}_3, \dots) = a_1e_1 + \sum_{i=2}^{\infty} \bar{a}_ie_i$  in  $M$ .



LEMMA 4.3. *Let  $\varphi : x_j R \rightarrow M_j$  be a monomorphism and let  $\varphi(x_j) = b_1 e_1 + \bar{b}_2 e_2 + \dots + \bar{b}_j e_j$ . Then  $b_1 R = x_j R = b_j R$ . Moreover,  $r \cdot \text{ann}(\bar{b}_1) = r \cdot \text{ann}(\bar{b}_j)$ .*

PROOF. Notice first of all that  $\varphi(x_1 R) = x_1 e_1 R$ . Hence  $\text{Soc}(\varphi(x_j R)) = x_1 e_1 R$ . It follows that  $\pi_1 \circ \varphi$ , the composition of  $\varphi$  with the projection  $\pi_1$  of  $M_j$  onto  $E$ , is one to one, for if  $\pi_1 \circ \varphi(x) = 0$ , then  $\varphi(x) \in \sum_{i=2}^j \bar{x}_i R$ . If  $\varphi(x) \neq 0$  then  $\text{Soc}(\varphi(x)R) \subset \sum_{i=2}^j \bar{x}_i R$ , while on the other hand  $\varphi(x)R \subset \varphi(x_j R)$ , and hence  $\text{Soc}(\varphi(x)R) = x_1 e_1 R$ , a contradiction. We conclude that  $\varphi(x) = 0$  and therefore, since  $\varphi$  is one to one,  $x = 0$ . Consequently,  $\pi_1 \circ \varphi(x_j R) = x_j R$ , which shows that indeed  $b_1 R = x_j R$ , as claimed. Now by definition of  $M_j$ ,  $\bar{b}_1 = \bar{b}_2 + \dots + \bar{b}_j$ . We conclude that  $b_j \notin x_{j-1} R$ . Thus  $x_j R = b_j R$ . Having shown that  $\pi_1 \circ \varphi$  is one to one, it follows that  $r \cdot \text{ann}(b_1 R) \subset [x_1 R : b_j R]$ . So  $\bar{b}_j R$  is a homomorphic image of  $b_1 R$  under the map given by  $b_1 \mapsto \bar{b}_j$ . Since  $\bar{b}_j R$  is of length  $j - 1$ , the kernel of the above map must be  $x_1 R$ . We therefore conclude that  $r \cdot \text{ann}(\bar{b}_1 R) = r \cdot \text{ann}(\bar{b}_j R)$ , as claimed.

THEOREM 4.4. *Let  $R$  be a bounded hnp ring having an indecomposable injective right  $R$ -module  $E$  with periodicity  $\geq 2$ . Then there exists a weakly-injective module  $M$  which does not admit a decomposition of the type described in Theorem 4.1 (3).*

PROOF. We shall prove that  $M$ , as defined in the remarks preceding Lemma 4.3, is a weakly-injective module, but that  $E$  does not embed in  $M$ . Consequently,  $M$  does not have a decomposition as described in the statement of the theorem. Notice first of all that  $\text{Soc}(M) \cong x_1 R \oplus \bar{x}_2 R \oplus \bar{x}_3 R \oplus \dots$ , since  $x_1 e_1 \in M$  and the set  $\{(\bar{x}_2 e_2 - \bar{x}_2 e_3)R, (\bar{x}_2 e_2 - \bar{x}_2 e_4)R, (\bar{x}_2 e_2 - \bar{x}_2 e_5)R, \dots\}$  of submodules of  $M$  constitutes an independent family of simple submodules of  $M$  each isomorphic to  $\bar{x}_2 R$ . It follows that  $E(M) \cong E \oplus \bar{E} \oplus \bar{E} \oplus \dots$ . Next, we show that  $M$  is weakly-injective. Let  $N$  be a finitely generated submodule of  $E(M)$ . Then  $N = y_1 R \oplus y_2 R \oplus \dots \oplus y_n R$ , where each  $y_i R$  is uniserial. If each  $y_i R$  has socle isomorphic to  $\bar{x}_2 R$ , for  $i = 1, \dots, n$ , then there exists  $j_i \in \mathbb{Z}^+$  such that  $y_i R \cong \bar{x}_{j_i} R \subset \bar{E}$ . Let  $j = \max\{j_i | i = 1, \dots, n\}$ . The submodules of  $M$ ,

$$(2) \quad (\bar{x}_{j_1} e_j - \bar{x}_{j_1} e_{j+1})R \cong y_1 R, \quad (\bar{x}_{j_2} e_{j+2} - \bar{x}_{j_2} e_{j+3})R \cong y_2 R, \quad \dots$$

$$\text{and } (\bar{x}_{j_n} e_{j+2(n-1)} - \bar{x}_{j_n} e_{j+2n-1})R \cong y_n R,$$

are an independent family whose sum is isomorphic to  $N$ . On the other hand, if for some  $i$ ,  $\text{Soc}(y_i R) \cong x_1 R$ , then, for some  $l \in \mathbb{Z}^+$ ,  $y_i R \cong x_l R$ . So, replace the corresponding submodule of  $M$  in (2) by  $(x_l e_1 + \bar{x}_l e_l)R \cong x_l R$ . Once again this yields an independent family of submodules whose sum is isomorphic to  $N$ . In light of Corollary 2.3, this concludes our proof of the weak-injectivity of  $M$ . Next we show that  $E$  is not embeddable in  $M$ . Assume on the contrary that  $\varphi : E \rightarrow M$  is

an embedding. We first observe that  $\varphi(x_1 R) = x_1 e_1 R$ . Similarly as in Lemma 4.3, if  $\pi_1$  is the projection of  $M$  onto  $E$ ,  $\pi_1 \circ \varphi$  is one to one. We obtain that for every  $j \in \mathbb{Z}^+$ , if  $\varphi(x_j) = a_1 e_1 + \bar{a}_2 e_2 + \cdots + \bar{a}_k e_k$ , with  $\bar{a}_k \neq 0$ , then (i)  $a_1 R = x_j R$ , (ii)  $k \geq j$ , and (iii) there exists  $l \in \mathbb{Z}$  such that  $j \leq l \leq k$  and  $a_l \notin x_{j-1} R$ . Let  $\varphi(x_2) = b_1 e_1 + \bar{b}_2 e_2 + \cdots + \bar{b}_k e_k$ , with  $\bar{b}_k \neq 0$  and consider then  $\varphi(x_{k+1}) = c_1 e_1 + \bar{c}_2 e_2 + \cdots + \bar{c}_t e_t$ , say. As observed above,  $t \geq k + 1$  and there exists  $l \in \mathbb{Z}^+$  such that  $k + 1 \leq l \leq t$  and  $c_l \notin x_k R$ . Define a map  $\varphi' : x_{k+1} R \rightarrow M_{k+1}$  via  $\varphi'(x_{k+1} r) = c_1 r e_1 + \bar{c}_2 r e_2 + \cdots + \bar{c}_k r e_k + \sum_{i=k+1}^t \bar{c}_i r e_{k+1}$ . Since  $\pi_1 \circ \varphi' = \pi_1 \circ \varphi$  is one to one, we conclude that  $\varphi'$  is also one to one. Applying Lemma 4.3, we get that  $r \cdot \text{ann}(\bar{d}) = r \cdot \text{ann}(\bar{c}_1)$ , where  $\bar{d} = \sum_{i=k+1}^t \bar{c}_i r$ . On the other hand, there exists  $y \in R$  such that  $x_{k+1} y = x_2$ . Hence  $\varphi'(x_{k+1} y) = \varphi'(x_2)$ . This implies that  $\bar{d} y = 0$  and therefore  $c_1 y \in x_1 R$ . However, since  $x_1 e_1 R \subset \varphi'(x_{k+1} R)$ , we would then get that  $\bar{b}_2 e_2 + \cdots + \bar{b}_k e_k = \bar{c}_2 e_2 y + \cdots + \bar{c}_k e_k y \in \varphi'(x_{k+1} R)$ . But  $\text{Soc}(\varphi'(x_{k+1} R)) = x_1 e_1 R$  and therefore we get  $\bar{b}_2 e_2 + \cdots + \bar{b}_k e_k = 0$ , a contradiction to the facts that  $k \geq 2$  and  $\bar{b}_k \neq 0$ . Thus, we conclude that  $E$  is not embeddable in  $M$ .

**THEOREM 4.5.** *Let  $R$  be a bounded hnp ring. Then the following conditions are equivalent:*

- (1) *Every uniform  $R$ -module has periodicity one.*
- (2) *Every weakly-injective  $R$ -module  $M$  has a decomposition  $M = B \oplus C$  such that  $B$  is injective and  $C$  satisfies that if a simple module  $S$  embeds in  $C$  then the module  $N_S$  embeds in  $C$ .*

**PROOF.** Apply Theorems 4.1 and 4.4

## References

- [1] A. H. Al-Huzali, S. K. Jain and S. R. López-Permouth, 'Rings whose cyclics have finite Goldie dimension', *J. Algebra* **153** (1992), 37–40.
- [2] D. Eisenbud and P. Griffith, 'Serial rings', *J. Algebra* **17** (1971), 389–400.
- [3] D. Eisenbud and J. C. Robson, 'Hereditary noetherian prime rings', *J. Algebra* **16** (1970), 86–104.
- [4] ———, 'Modules over Dedekind prime rings', *J. Algebra* **16** (1970), 67–85.
- [5] J. S. Golan and S. R. López-Permouth, 'QI-filters and tight modules', *Comm. Algebra* **19** (8) (1991), 2217–2229.
- [6] S. K. Jain and S. R. López-Permouth, 'Rings whose cyclics are essentially embeddable in projectives', *J. Algebra* **128** (1990), 257–269.
- [7] S. K. Jain, S. R. López-Permouth and S. Singh, 'On a class of QI-rings', *Glasgow Math. J.* **34** (1992), 75–81.
- [8] T. H. Lenagan, 'Bounded hereditary noetherian prime rings', *J. London Math. Soc.* **6** (1973), 241–246.

[11] Weakly-injective modules over hereditary noetherian prime rings 297

- [9] H. Marubayashi, 'Modules over bounded Dedekind prime rings', *Osaka J. Math.* **9** (1972), 95–110.
- [10] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings* (Wiley, New York, 1987).
- [11] K. M. Rangaswami and Vanaja, 'Quasi projectives in abelian categories', *Pacific J. Math.* **48** (1972).
- [12] S. Singh, 'Quasi-injective and quasi-projective modules over hereditary noetherian prime rings', *Canad. J. Math.* **26** (1974), 1173–1185.
- [13] ———, 'Modules over hereditary noetherian prime rings', *Canad. J. Math.* **27** (1975), 867–883.
- [14] ———, 'Modules over hereditary noetherian prime rings, II', *Canad. J. Math.* **28** (1976), 73–82.

Ohio University  
Athens, Ohio 45701