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Rings Whose Cyclics Have Finite Goldie Dimension

A. H. AL-HUZALI, S. K. JAIN, AND S. R. LÓPEZ-PERMOUTH

Department of Mathematics, Ohio University, Athens, Ohio 45701

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A module M is said to be weakly injective if every finitely generated submodule of its injective hull $E(M)$ is contained in a submodule X of $E(M)$ isomorphic to M. We prove that a ring R satisfies the property that every cyclic right R-module has finite Goldie dimension if and only if every direct sum of (weakly) injective right R-modules is weakly injective. This is analog to the well-known characterization of right Noetherian rings as those for which direct sums of injective right modules are injective.

PRELIMINARIES

The notation and terminology of this paper are standard, as may be found in [1]. A ring R is said to be a right q.f.d. ring if every cyclic right R-module has finite Goldie dimension. Right q.f.d. rings have been studied in [2, 5, 6]. If we were following [5], right q.f.d. rings would be called right FGS-rings. The reasons for the different terminologies are explained in the following proposition, which compiles results from [2, 5, 6]. Here and throughout the paper the notation $A \subset' B$ will mean that A is an essential submodule of the module B.

PROPOSITION. *For a ring R the following conditions are equivalent*:

(1) *Every cyclic right R-module has finite Goldie dimension.*

(2) *Every cyclic right R-module has finitely generated* (*possibly zero*) *socle.*

(3) *For any cyclic right R-module C and for every properly ascending chain* $A_1, \subsetneq A_2 \subsetneq \cdots$ *of submodules of C there exists* $n > 0$ *such that* $A_n \subset' A_m$ *for all* $m \geqq n$.

(4) *Every right ideal N of R contains a finitely generated right ideal T such that N/T has no maximal submodules.*

(5) *Every finitely generated right R-module has finite Goldie dimension.*

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The class of q.f.d. rings contains all rings with right Krull dimension. So, in particular, every right noetherian ring is right q.f.d.

Given two right modules M and N, we say that M is weakly N-injective if for every homomorphism $\varphi: N \to E(M)$ there exists a submodule $X \subseteq E(M)$ which is isomorphic to M satisfying $\varphi(N) \subseteq X$. Clearly, M is weakly N-injective if and only if it is weakly (N/K)-injective for every submodule K of N. So, M is weakly R^n-injective if and only if it is weakly N-injective for every module N generated by n elements and, in particular, M is weakly R-injective if and only if it is weakly N-injective with respect to every cyclic module N.

We say that a right module M is weakly injective if it is weakly N-injective for every finitely generated right module N. Weakly injective modules are closed under finite direct sums and essential extensions (See [3, 4]).

One can easily see that a module M is weakly R-injective (weakly injective) if and only if for every element $q \in E(M)$ (finitely generated submodule $N \subset E(M)$) there exists a submodule $X \subset E(M)$ which is isomorphic to M such that $q \in X$ $(N \subset X)$.

The concept of weak relative injectivity of modules was introduced in [3] to characterize semiperfect rings whose every cyclic right module is embeddable as an essential submodule of a projective module. Later on, among other results, rings over which all right modules are weakly injective are characterized in [4]. The purpose of this paper is to characterize rings over which direct sums of weakly injective right modules are again weakly injective. It turns out that such rings are precisely the right q.f.d. rings.

The Result

THEOREM. *For a ring R the following conditions are equivalent*:

(1) *R is a right q.f.d. ring.*

(2) *Every direct sum of injective right R-modules is weakly injective.*

(3) *Every direct sum of weakly injective right R-modules is weakly injective.*

(4) *Every direct sum of weakly injective right R-modules is weakly R-injective.*

(5) *Every direct sum of indecomposable injective right R-modules is weakly R-injective.*

Proof. We prove first that (1) $\Rightarrow$ (2).

Consider $M = \oplus \sum_{i \in \Lambda} E_i$, where, for every $i \in \Lambda$, E_i is an injective right R-module.

Let N be a finitely generated submodule of $E(M)$. By the hypothesis, N contains as an essential submodule a direct sum of uniform submodules $U_1 \oplus U_2 \oplus \cdots \oplus U_k$. Since $M \subset' E(M)$ there exists $0 \neq q_i \in U_i \cap M$. So, $\oplus \sum_{i=1}^{k} q_i R$ is contained in a finite subsum $E_{i_1} \oplus E_{i_2} \oplus \cdots \oplus E_{i_t}$ of M. This implies that $E_{i_1} \oplus E_{i_2} \oplus \cdots \oplus E_{i_t}$ contains an injective hull E of $\oplus \sum_{i=1}^{k} q_i R$. Since E is injective and contained in M, we may write $M = E \oplus K$, for some submodule K of M. On the other hand, let $E(N)$ be an injective hull of N inside $E(M)$. Then $E(N) = \oplus \sum_{i=1}^{k} E(U_i) = \oplus \sum_{i=1}^{k} E(q_i R) \cong E$. Since $\oplus \sum_{i=1}^{k} q_i R \subset' E(N)$, it follows that $E(N) \cap K = 0$. So, let $X = E(N) \oplus K \cong E \oplus K = M$. Then $N \subset X$, proving our claim.

Condition (2) obviously implies (5). Once we have shown that (5) implies (1) we will have the equivalence of (1), (2), and (5). Let us proceed now to prove that (5)$\Rightarrow$(1). Let R/I be a cyclic right R-module. If $\mathrm{Soc}(R/I) = 0$, we are done. Suppose $M = \mathrm{Soc}(R/I) \neq 0$.

Write $M = \oplus \sum_{i \in \Lambda} S_i$ as a direct sum of simple R-modules S_i. We show that M is finitely generated.

Clearly, $E(M) = E(\oplus \sum_{i \in \Lambda} S_i) = E(\oplus \sum_{i \in \Lambda} E(S_i)) = \hat{E}$, say.

By hypothesis $\oplus \sum_{i \in \Lambda} E(S_i)$ is weakly R-injective. Hence it is weakly (R/I)-injective. Consider now the diagram

$$\begin{array}{ccc} M = \oplus \sum_{i \in \Lambda} S_i & \xrightarrow{\varphi} & \hat{E} = E\left(\oplus \sum_{i \in \Lambda} E(S_i)\right) \\ \big\downarrow \lambda & & \\ \dfrac{R}{I} & & \end{array}$$

where φ and λ are inclusion R-homomorphisms. By the injectivity of $\hat{E}$, there exists $\hat{\varphi}$ such that $\hat{\varphi}\lambda = \varphi$. Further, since $\oplus \sum_{i \in \Lambda} E(S_i)$ is weakly (R/I)-injective, there exists $X \subset \hat{E}$ such that $\hat{\varphi}(1 + I) \in X \cong \oplus \sum_{i \in \Lambda} E(S_i)$.

Hence there exists a finite subset Γ of Λ and an independent family of submodules $\{X_i\}_{i \in \Gamma}$ such that $\hat{\varphi}(1 + I) \in \oplus \sum_{i \in \Gamma} X_i$, and $X_i \cong E(S_i)$ for all $i \in \Gamma$.

Also, we have $M = \varphi(M) \subset \hat{\varphi}(R/I) = \hat{\varphi}(1 + I)R \subset \oplus \sum_{i \in \Gamma} X_i$.

Since each X_i is uniform, M has finite uniform dimension and is therefore finitely generated.

Let us show next that (2)$\Rightarrow$(3). Consider then the module $M = \oplus \sum_{i \in \Lambda} M_i$, a direct sum of weakly injective modules M_i, $i \in \Lambda$. Let N be a finitely generated submodule of $E(M)$. By (2) the direct sum of injectives $\oplus \sum_{i \in \Lambda} E(M_i)$ is weakly injective. Also,

$$M \subset' \oplus \sum_{i \in \Lambda} E(M_i) \subset' E(M).$$

Hence, by (2), there exists a submodule $Y \subset E(M)$ such that $N \subset Y$ and $Y \cong \bigoplus \sum_{i \in \Lambda} E(M_i)$. Write $Y = \bigoplus \sum_{i \in \Lambda} E(Y_i)$ such that $M_i \cong Y_i$ for all $i \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma \subset \Lambda$ such that $N \subset \bigoplus \sum_{i \in \Gamma} E(Y_i) = E(\bigoplus \sum_{i \in \Gamma} Y_i)$. Since the Y_i's are weakly injective, the finite sum $\bigoplus \sum_{i \in \Gamma} Y_i$ is weakly injective and therefore there exists $X_1 \cong \bigoplus \sum_{i \in \Gamma} Y_i \cong \bigoplus \sum_{i \in \Gamma} M_i$ such that $N \subset X_1 \subset E(\bigoplus \sum_{i \in \Gamma} Y_i)$. But then $N \subset X_1 \oplus \bigoplus \sum_{i \notin \Gamma} Y_i = X \cong M$, proving our claim. The implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are trivial, concluding our proof of the theorem.

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