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ON THE WEAK RELATIVE-INJECTIVITY OF RINGS AND MODULES

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1. INTRODUCTION

This paper is about two strongly related topics, right CEP-rings and weakly-injective rings and modules. Rings whose cyclic modules are embeddable essentially in direct summands and rings whose cyclics are embeddable essentially in projectives have been studied in [6], [7] and [8]. Following [6], we refer to the latter as right CEP-rings. We propose refering to the former as right CES-rings. In the above mentioned references, semiperfect CES-rings are characterized as follows:

Theorem A. Let R be a semiperfect ring. R is right CEP if and only if R is right artinian and every summand of R is weakly R-injective.

Theorem B. The following statements about a ring R are equivalent:

(i) R is semiperfect and right CES,

(ii) every ring homomorphic image of R is right CES,

(iii) every ring homomorphic image of R is right CEP,

(iv) R is of one of the following types:

(a) R is uniserial as a right module,

(b) R is an $n \times n$ matrix ring over a right self-injective ring of the type in (a), or

(c) R is a direct sum of rings of types (a) or (b).

One is left to wonder about the necessity of the assumption that R is semiperfect. In this paper we show that right CES-rings and right CEP-rings are necessarily semiperfect in many special cases including when R is semiprime, right nonsingular, right semi-hereditary, right self-injective, or one-sided Noetherian.

The concept of weak relative-injectivity of modules as mentioned in Theorem A was first introduced in [7]. Its study has been furthered in [8] and [5]. The second reference deals with the somewhat more general concept of tight modules. Weak relative-injectivity of modules is closed under finite direct sums but it notably fails to be inherited by direct summands. The same holds true for tightness. We provide examples of a countably infinite collection of modules none of which is weakly-injective whose sum is weakly-injective. Also, we investigate when the condition of weak-injectivity on (the summands of) a semiperfect ring is equivalent to the injectivity of the ring. Our results include the fact that a semiperfect ring R is right self-injective if and only if each summand of R is weakly-injective and the Jacobson radical of R coincides with its right singular ideal. Also, we show that a right or left perfect ring is self-injective if and only if it is weakly-injective.

Throughout this paper R is a ring with 1 and all modules are right and unital unless otherwise specified. As usual, $J(M)$, $Z(M)$ and $E(M)$ denote respectively the Jacobson radical, the singular submodule and the injective hull of a module M. Any term not defined here may be found in a standard reference such as [1].

2. CEP-RINGS

Given two modules M and N, we say that M is weakly N-injective if for every map $f : N \to E(M)$, one may find a submodule X of $E(M)$ which is isomorphic to M and satisfies that the image of f lies in X. We also say that M is N-tight if every homomorphic image of N which is embeddable in $E(M)$ is also embeddable in M. Obviously, every weakly N-injective module is N-tight as well. If a module is weakly R^n-injective (R^n-tight) for all positive integer n, we say that M is weakly-injective (tight). When M is cyclic one easily sees that M is weakly-injective (tight) if and only if it is weakly R^2-injective (R^2-tight).

The exact relationship between weak-injectivity and tightness is given by the following lemma from [8].

2.1 Lemma. Given two right modules M and N, M is weakly N-injective if and only if for every submodule Q of N and for every monomorphism $\sigma : N/Q \to E(M)$

(i) there exists a monomorphism $\sigma' : N/Q \to M$ and

(ii) for every complement K of $\sigma'(N/Q)$ in M, there exists a submodule K' of $E(M)$ such that $K' \cap \sigma(N/Q) = 0$ and $K' \cong K$.

Proof. See [8]. □

2.2 Corollary. A uniform module M is weakly N-injective if and only if it is N-tight.

Proof. Obvious. □

In view of this corollary, the following characterization of right CEP-rings follows from Proposition 1.12 in [8].

2.3 Proposition. A semiperfect ring R is right CEP if and only if R is right artinian and every projective indecomposable right R module is uniform and tight.

Proof. Immediate from the discussion. □

If one assume that every right module is tight, then, in particular, every uniform right module is weakly-injective. Also, under this hypothesis, completely reducible right R-modules would have to be injective which implies that R is right Noetherian. In light of Theorem 2.5 in [8], we arrive to the following proposition.

2.4 Proposition. A ring R is right weakly-semisimple (i.e., every right modules is weakly-injective) if and only if any one of the following equivalent conditions holds:

(i) every right R-module is tight;

(ii) R is right Noetherian and every finitely generated right R-module is tight;

(iii) R is right Noetherian and every cyclic right R-module is tight; or

(iv) R is right Noetherian and every uniform cyclic right R-module is tight.

Proof. Every right weakly-semisimple ring satisfies (i). By the above discussion, (i) implies (ii). Clearly, (ii) implies (iii) and (iii) implies (iv). By Corollary 2.2, (iv) is equivalent to "R is right Noetherian and every uniform cyclic right R-module is weakly R-injective" ; hence by Theorem 2.5 in [8], (iv) implies that R is weakly-semisimple. □

The following theorems show that rings whose cyclics are embeddable in free modules (thus right CEP-rings) are semiperfect under any of several additional hypotheses. Some of these results are implicit in [2].

2.5 **Theorem.** A ring R for which every cyclic right module embeds in a free is semisimple artinian if R is semiprime, right nonsingular or right semi-hereditary.

Proof. Suppose R is not semisimple artinian and let M be a maximal essential right ideal. By hypothesis, R/M embeds in a free module F via a monomorphism $\varphi : R/M \to F$. Easily one concludes that F must be a finite sum $F = R \oplus \cdots \oplus R$. Since R is simple one of the projection maps $\pi : F \to R$ satisfies that $\pi\ \varphi(R/M) \cong R/M$. In the case when R is semiprime, $\varphi(R/M) = eR$, where $e^2 = e$ is an idempotent. This implies that $(1-e)R$, the right annihilator of e, equals M. This contradicts the assumption that M is essential. We conclude that when R is semiperfect R is semisimple-artinian. Consider now the case when R is right nonsingular. In this case, R/M is isomorphic to a single right idea aR of R, with $r(a) = M$. Since M is assumed to be semisimple, one concludes that a belongs to the right singular ideal $Z(R)$. This contradicts the nonsingularity of R. Once again, we conclude that R must be semisimple artinian. Lastly, suppose R is right semi-hereditary (*i.e.*, every principal right ideal is projective). Since R/M is isomorphic to a simple right ideal aR or R and aR is projective, one concludes that M splits in R, contradicting its essentiality. Therefore, if R is semi-hereditary, it must be semisimple artinian. □

2.6 **Theorem.** A ring R for which every cyclic right module embeds in a free is artinian whenever R is right self-injective or one-sides Noetherian.

Proof. If R is right self-injective the result follows Theorem 3.3 in [7], since every right self-injective ring is right QF-3. Alternatively, the result follows from the fact that a right self-injective ring satisfying the hypothesis is right PF and hence semiperfect [11]. When R is left Noetherian it satisfies the ascending chain condition (ACC) on annihilator left ideals and, equivalently, the descending chain condition (DCC) on annihilator right ideals. The result follows since, under our hypothesis, every right ideal is an annihilator right ideal. Finally, assume that R is right Noetherian. By Lemma 4 in [10], R^n has essential right socle for every positive integer n. Since every cyclic embeds in some R^n we conclude that every right module has nonzero socle. It follows that R is left perfect and hence, by Hopkin's theorem, right artinian. □

A right Noetherian ring for which every cyclic right module is an annihilator does not need to be right artinian [4].

3. WEAK-INJECTIVITY

As mentioned in the introduction, any finite direct sum of weakly-injective (tight) modules is weakly-injective (tight) but a direct summand of a weakly-injective (tight) module may not be weakly-injective (tight). Indeed, any module M over an arbitrary ring is a direct summand of a tight module $T = M \oplus H$, if we take H to be the direct product of an infinite number of copies of the injective hull of M. Similarly, any module M over a right Noetherian ring R is a summand of a weakly-injective module $W = M \oplus S$. Simply take S to be a direct sum of infinitely many copies of the injective hull of M. In order to illustrate further how weak-injectivity is not transferred down to summands, we present next an example of a countable family of non-weakly-injective modules whose sum is weakly-injective.

3.1 Example. Let $M_i = \mathbf{Z}/2^i\mathbf{Z}$. The (external) direct sum $M = \oplus\sum_{i=1}^{\infty} M_i$ is a weakly-injective but for all i, M_i is not weakly-injective.

Proof. Since M_i is finite while its injective hull $E(M_i) = \mathbf{Z}(2^\infty)$ is not, obviously M_i is not even tight. As usual, we shall identify $\mathbf{Z}/2^i\mathbf{Z}$ with the submodule $\{0, \frac{1}{2^i}, \frac{2}{2^i}, \cdots, \frac{2^{i-1}}{2^i}\}$ of $\mathbf{Z}(2^\infty)$. Then, the $M_i's$ constitute an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ whose union is $\mathbf{Z}(2^\infty)$. Let $x_1, \cdots, x_n \in E(M)$. Since $E(M)$ is a countably infinite sum $\oplus \sum_{i=1}^{\infty} \mathbf{Z}(2^\infty)$ of copies of $\mathbf{Z}(2^\infty)$, $x_1, \cdots, x_n$ belong to a finite sub-sum $N = \oplus \sum_{i=1}^{t} \mathbf{Z}(2^\infty)$. Since each copy of $\mathbf{Z}(2^\infty)$ is a union of the chain $M_1 \subseteq M_2 \subseteq \cdots$, one can choose inside the i^{th} copy of $\mathbf{Z}(2^\infty)$ a submodule isomorphic to M_{n_i} in such a way that $M_{n_i} \not\cong M_{n_j}$, when $i \neq j$ and $x_1, \cdots x_n \in M_{n_1} \oplus \cdots \oplus M_{n_t}$. Out of the remaining summands of M pick copies of the $M_i's$ whose isomorphism classes are not represented in $\{M_{n_1}, \cdots, M_{n_t}\}$. the sum of these submodules plus the $M'_{n_i}s$ is a submodule X of $E(M)$ which is isomorphic to M and satisfies that $x_1, \cdots, x_n \in X$, proving that M is weakly-injective. □

It is known that, for many rings, weak-injectivity is quite a distinct property from injectivity. For example, in [8] it has been shown that a domain R is weakly-injective if and only if it is two-sided Ore. However, in that same reference we see that if a ring R is right artinian, then R is right weakly-injective if and only if it is right self-injective (*i.e.*, R is quasi-Frobenius). Our next theorems are concerned with then the condition of weak-injectivity on the summands of a semi-perfect ring R is equivalent to the self-injectivity of R. For the remainder of the paper the semi-perfect ring R may be written as $R = \oplus \sum_{i=1}^{n} e_i R$ where End $(e_i R)$ is local for $i = 1, \cdots, n$.

3.2 Theorem. Let R be a semiperfect ring then

(i) R is self-injective if every summand of R is weakly-injective and R has nil Jacobson radical $J(R)$, and

(ii) R is self-injective if and only if every summand of R is weakly-injective and the Jacobson radical $J(R)$ coincides with the right singular ideal $Z_r(R)$.

Proof.

(i) Consider a projective indecomposable module eR with $e^2 = e$. By its weak-injectivity, we know that since e, $x \in E(eR)$ there exists $X \cong eR$ such that e, $x \in X$. This means that X is local, hence if $eR \neq X$ we must have that e belongs to the Jacobson radical $J(X)$ of X. So, there exists an embedding φ : eR $\rightarrow eJ$. Let $\varphi(e) = ea = a$. Let $n \in \mathbf{Z}$ be the smallest such that $a^n = 0$. Then $\varphi(ea^{n-1}) = a \cdot a^{n-1} = 0$, but since φ is one-to-one $ea^{n-1} = a^{n-1} = 0$, a contradiction. We, therefore, conclude that $eR = X$ and hence $x \in eR$. Since x was chosen arbitrarily, we conclude that $eR = E(eR)$ is indeed injective.

(ii) It is well known that a self-injective ring satisfies that $J(R) = Z_r(R)$ and obviously each summand of R must be (weakly-) injective. Conversely, consider an indecomposable projective module eR, with $e^2 = e$. Arguing as before, for arbitrary $x \in E(eR)$, e and x belong to a local submodule X of $E(eR)$ which is isomorphic to eR. If $eR \neq X$ we get an embedding φ of eR into eJ. This leads to a contradiction for if $\varphi(e) = a \in J(R) = Z_r(R)$, then the right annihilator of a, $r(a)$, must be essential. However, $r(a) = r(e) = (1-e)R$. so $1-e$ is an element of an essential right ideal of R. This leads to a contradiction. Therefore, we conclude that $x \in eR$ and then $eR = E(eR)$, as desired. □

The following theorem concerns the weak-injectivity of a right or left perfect ring.

3.3 Theorem. A right or left perfect ring R is self-injective if and only if it is weakly-injective.

Proof. Assume that R is left perfect. This is equivalent to saying that R satisfies the DCC on principal right ideals. Let 1, $x \in E(R)$, and let X be a submodule of $E(R)$, isomorphic to R such that 1, $x \in X$. If $R \neq X$ then R is embeddable as a proper principal right ideal of itself. This would yield an infinite decreasing sequence of principal right ideals, a contradiction. Hence $R = X$ and, therefore, since x was picked arbitrarily one concludes that $R = E(R)$ is self-injective. On the other hand, assume that R is right perfect. A result of Jonah [9] states that this is equivalent to every right module of R satisfying the ACC on cyclic submodules. In particular, $E(R)$ satisfies such ACC.

As before let $x \in E(R)$ and let X be a submodule of $E(R)$, isomorphic to R such that 1 and x belong to X. If $R \neq X$ then X is a cyclic submodule of $E(R)$ which contains R properly. Yet, X is isomorphic to R via an isomorphism $\varphi : R \rightarrow X$, say. Extend φ to an isomorphism $\hat{\varphi} : E(R) \rightarrow E(X) = E(R)$. Then $\hat{\varphi}(X)$ is a cyclic submodule of $E(R)$ containing X properly. Continuing this way one gets an infinite ascending chain of cyclic submodules of $E(R)$, a contradiction. Thus, $R = X$ and since x was chosen arbitrarily $R = E(R)$ is self-injective. □

3.4 Corollary. A semiprimary ring R is self-injective if and only if it is weakly-injective.

Proof. Semiprimary rings are right and left perfect.

3.5 Corollary. A right or left artinian ring R is quasi-Frobenius if and only if it is weakly-injective.

Proof. Right or left artinian rings are semiprimary.

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