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A CHARACTERIZATION OF UNISERIAL RINGS VIA CONTINUOUS AND DISCRETE MODULES

S. K. JAIN, S. R. LÓPEZ-PERMOUTH AND S. TARIQ RIZVI

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Abstract

It is well-known that quasi-Frobenius rings are characterized by the property that all propective right modules are injective, as well as by the property that all injective right modules are projective. Similarly, either the property that every quasi-projective is quasi-injective or that every quasi-injective is quasi-projective characterizes uniserial rings. Oshiro recently has given similar characterizations for generalized uniserial rings. The purpose of this paper is to characterize rings for which continuous right modules are discrete. We show that these rings are precisely the uniserial rings. The property that every discrete module is continuous is also investigated.

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1. Introduction

It is well-known that quasi-Frobenius rings are characterized by the property that all projective right modules are injective, as well as by the property that all injective right modules are projective ([2], [3]). Similarly, either the property that every quasi-projective characterizes uniserial rings ([1], [5]). Oshiro has given similar characterizations of generalized uniserial rings [16]. The purpose of this paper is to characterize rings for which continuous right modules are discrete (that is, dual-continuous). We show that these rings are precisely the uniserial rings (Theorem 3.3). For a ring R the property that every discrete right R-module is continuous is also equivalent to R being uniserial when either R is right perfect or R is semiperfect with finitely generated radical (Theorem 3.8). A comprehensive treatment of discrete and continuous modules may be found in a forthcoming book by Mohamed-Mueller [15].

2. Definitions and preliminaries

Throughout this paper all rings have 1 and modules are right unital unless otherwise stated. The Jacobson radical of a ring R is denoted by $J(R)$ or simply by J. For any R-module M, $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, and $E(M)$ denote its radical, socle, and injective hull respectively. A module M is said to be valuation (also called uniserial) if its submodules are linearly ordered under inclusion. A ring R is said to be right serial if it is a direct sum of valuation right ideals, and serial if it is both left and right serial. An artinian ring in which each one-sided ideal is principal is called a uniserial ring. It is well-known that a uniserial ring is serial. Artinian serial rings are called generalized uniserial.

For R-modules N and M, we write $N \subset' M$ ($N \overset{\oplus}{\subset} M$) to denote that N is essential in M (N is a summand of M). For a right module M, consider the following conditions:

(C_1) For every submodule N of M there exists $M_1 \overset{\oplus}{\subset} M$ with $N \subset' M_1$.

(C_2) For any summand M' of M, every exact sequence $0 \to M' \to M$ splits.

(C_3) If M_1 and M_2 are summands of M with $M_1 \cap M_2 = 0$, then $M_1 + M_2$ is a summand of M.

A module M is called continuous (quasi-continuous) if it satisfies (C_1) and (C_2) ((C_1) and (C_3)). These conditions are dualized as follows.

(D_1) For every submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subset N$ and $M_2 \cap N$ is small in M_2.

(D_2) For any summand M' of M, every exact sequence $M \to M' \to 0$ splits.

(D_3) If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M.

A module M is called discrete (quasi-discrete) if M satisfies (D_1) and (D_2) ((D_1) and (D_3)). Discrete (quasi-discrete) modules, have also been called dual continuous (respectively quasi-dual continuous) ([13], [14]). It is easy to see that (C_2) implies (C_3) and (D_2) implies (D_3). Therefore, the following hierarchy exists:

$$\text{Injective} \Rightarrow \text{quasi-injective} \Rightarrow \text{continuous} \Rightarrow \text{quasi-continuous},$$

and

Projective $\Rightarrow$ quasi-projective $\nRightarrow$ discrete $\Rightarrow$ quasi-discrete.

Notice that the condition (D_1) is equivalent to

(D_1') every submodule N of M can be written as $N = N_1 + N_2$ where $N_1 \subset 4^+M$, and N_2 is small in M.

We list below some well-known results which will be used frequently.

2.1. Proposition [13, Propositions 1.2 and 1.5]. *If $M = \bigoplus \sum M_i$ is (quasi-) discrete then M_i is (quasi-) discrete and M_j-projective for $i = j$.*

2.2. Proposition [8, Proposition 1.7 and 1.12]. *If $M = \bigoplus \sum M_i$ is (quasi-) continuous, then each M_i is (quasi-) continuous and M_j-injective for $i = j$.*

2.3. Proposition [7, Proposition 3.1]. *If M is M_i-projective for all $i \in I$ and M is finitely generated then M is $\bigoplus \sum M_i$-projective.*

2.4. Proposition [13, Theorem 2.2]. *Every quasi-discrete module M has a decomposition $M = \bigoplus \sum M_i$, where each M_i is indecomposable. Further, this decomposition complements summands.*

2.5. Proposition [14, Theorem 2.3]. *A ring R is right (semi-) perfect if and only if every (finitely generated) quasi-projective R-module is discrete.*

2.6. Proposition ([5], Theorem 5.1) and ([1], Proposition 2.6). *A ring R is (generalized) uniserial if and only if every (indecomposable) quasi-projective right module is quasi-injective, if and only if every (indecomposable) quasi-injective right module is quasi-projective.*

2.7. Proposition [4, Theorem 25.4.2]. *If R is a generalized uniserial ring then each R-module is a direct sum of cyclic valuation modules.*

3. Main results

We start by stating two of our main theorems.

3.1. Theorem. *For a ring R, the following are equivalent:*

(1) *R is uniserial;*

(2) *every quasi-injective module is quasi-discrete;*

(3) *every quasi-injective module is discrete*;
(4) *every continuous module is quasi-discrete*;
(5) *every continuous module is discrete*;
(6) *every quasi-continuous module is quasi-discrete*;
(7) *every quasi-continuous module is discrete.*

3.2. THEOREM. *Let R be a semiperfect ring such that $J = J(R)$ is finitely generated. Then the following conditions are equivalent*:
(1) *R is a uniserial ring*;
(2) *every (quasi-) discrete R-module is quasi-injective*;
(3) *every (quasi-) discrete R-module is (quasi)-continuous*;
(4) *every finitely generated quasi-projective R-module is (quasi)-continuous*;
(5) *every quasi-projective R-module is (quasi-) continuous.*

Before we prove these theorems we prove a lemma.

3.3. LEMMA. *If R is a generalized uniserial ring then*
(i) *every quasi-discrete R-module is quasi-projective, and*
(ii) *every quasi-continuous R-module is quasi-injective.*

PROOF. Let M be an R-module. Since R is a generalized uniserial ring, we write $M = \bigoplus\sum_i M_i$, where each M_i is a cyclic valuation submodule.

(i) Assume M is quasi-discrete. Then by Proposition 2.1, M_i is M_j-projective for $i \neq j$. Also, since M_i is indecomposable, M_i is M_i-projective by Proposition 2.6. Furthermore, because M_i is cyclic, M_i is $M = \bigoplus\sum M_i$-projective. But then M is also M-projective, that is, M is quasi-projective as desired.

(ii) Assume now M is quasi-continuous.

This implies M_i is M_j-injective for $i \neq j$ (Proposition 2.2). Again as in (i), M_i is M_i-injective. Now, because R is noetherian, $E(M) = \bigoplus\sum_i E(M_i)$. Let $(\varphi_{ij}) \in \mathrm{End}(E(M))$ where $\varphi_{ij} \in \mathrm{Hom}(E(M_j), E(M_i))$, and let $\sum x_k \in \bigoplus\sum M_i$. Then $(\varphi_{ij})(\sum x_k) = \sum \varphi_{ik} x_k \in \bigoplus\sum M_i$, since M_i is M_k-injective. Hence $M = \bigoplus\sum M_i$ is quasi-injective.

PROOF OF THEOREM 3.1. Since any one of the statements (3) through (7) implies statement (2) and statement (1) implies all others (Lemma 3.3), all we need to show is that R is uniserial if each quasi-injective R-module is quasi-discrete.

Assuming (2), we have that R must be generalized uniserial [16, Theorem 2]. It then follows from Lemma 3.3 that every quasi-injective R module is quasi-projective, and hence R is uniserial.

PROOF OF THEOREM 3.2.

(1) $\Rightarrow$ (2). If M is (quasi-) discrete then M is quasi-projective (Lemma 3.3) and therefore M is quasi-injective since R is uniserial that (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Now M is finitely generated quasi-projective so M is discrete (Proposition 2.5), and hence M is continuous.

(4) $\Rightarrow$ (1). Since R is semiperfect, we can write $R = \bigoplus \sum e_i R$, where $e_i R$ is indecomposable and $e_i e_j = \delta_{ij} e_i$ for $i, j = 1, \dots, n$. Now, $(e_i R)/(e_i J^m)$ is a finitely generated quasi-projective module. Thus by hypothesis, $(e_i R)/(e_i J^m)$ is quasi-continuous and hence uniform. We proceed to show that $(e_i R)/(e_i J^m)$ is valuation and noetherian.

Set $J^\omega = \bigcap_{k=1}^\infty J^k$. Let $0 \neq A/e_i J^\omega \subset e_i R/e_i J^\omega$. There exists a positive integer k such that $A \not\subset e_i J^k$. Choose the smallest s such that $A \not\subset e_i J^s$. Therefore, $A \subset e_i J^{s-1}$. Furthermore, $\mathrm{Rad}(e_i J^{s-1}) = e_i J^s$, since R is semiperfect. Also, $(e_i J^{s-1})/(e_i J^s)$ is a completely reducible R/J- (and hence R-) module which must be simple since $e_i R/e_i J^s$ is uniform. This yields that $e_i J^s$ is the unique maximal submodule of $e_i J^{s-1}$ which is small since J and hence $e_i J^{s-1}$ are finitely generated. Thus $A + e_i J^s = e_i J^{s-1}$ implies $A = e_i J^{s-1}$. Therefore, every nonzero submodule of $e_i R/e_j J^\omega$ is of the form $(e_i J^s)/(e_i J^\omega)$. This yields that $e_i R/e_i J^\omega$ is noetherian and valuation. Therefore, R/J^ω is a right noetherian right serial ring. Moreover, by hypothesis, $R/J^\omega \times R/J^\omega$ is quasi-continuous. Therefore, by Proposition 2.2, R/J^ω is self-injective and so it is quasi-Frobenius. This implies that there exists $t > 0$ such that

$$J^t = J^{t+1} = \cdots .$$

But then, by the Nakayama Lemma, $J^t = 0$, which yields $J^\omega = 0$. Consequently, R is quasi-Frobenius. Furthermore, for any ideal A of R, R/A is also self-injective since $R/A \times R/A$ is quasi-continuous. This gives that R/A is quasi-Frobenius. Hence, R is uniserial [4, Proposition 25.4.6(B)].

Finally, (1) $\Rightarrow$ (5) follows from the fact that over uniserial rings quasi-projectives are quasi-injective, while (5) $\Rightarrow$ (4) is obvious. This completes the proof.

The following example shows that the hypothesis that J is finitely generated may not be removed from Theorem 3.2.

3.4. Example. Let F be a field. Let $\Omega = \{I \subset R^+ \cup \{0\} \mid I \text{ is well-ordered}\}$. Let T be the set of formal power series $\sum a_i x^i$ over F, with $I \in \Omega$. It is well known that for every nonzero ideal A of T, T/A is self-injective [11]. Let $R = T/xT$; then R is a local, non-noetherian, commutative ring satisfying the condition that every one of its homomorphic images injective. Since any finitely generated quasi-projective R-module M ect sum of copies of R/B, where B is an ideal of R [10], we obtain I is quasi-injective. Therefore, R is a non-uniserial ring satisfying

ie next theorem we trade the hypothesis of having a finitely generated for that of being perfect.

THEOREM. *For a ring R the following conditions are equivalent*:
R is uniserial;
very (quasi-) discrete R-module is quasi-injective and R is right perfect;
every (quasi-) discrete R-module is (quasi-)continuous and R is right

every quasi-projective R-module is (quasi-)continuous.

OF. Note (1) $\Rightarrow$ (2) follows by Lemma 3.3, (2) $\Rightarrow$ (3) is trivial and (4) follows from Proposition 2.5. Thus we only need to verify (4) $\Rightarrow$

ime (4). Let A be an ideal of R and S a direct sum of copies of Since S is quasi-projective, it is quasi-continuous. It follows from ition 2.2 and every free R/A module is injective. This implies that quasi-Frobenius and so R is uniserial.

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Ohio 45701

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