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Semiperfect rings whose proper cyclic modules are continuous [1])

By

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1. Introduction. Modules M with the properties: (P1) every submodule N of M is essential in a direct summand of M and (P2) every submodule N of M which is isomorphic to a direct summand of M is itself a direct summand, are called continuous modules. All injective or quasi-injective modules are clearly continuous. In this paper we characterize semiperfect rings whose proper cyclic modules are continuous. This characterization generalizes the results for semiperfect rings whose proper cyclic modules are injective [1] (or quasi-injective [2]).

2. Definitions and Notation. The Jacobson radical of a ring R will be denoted by $J(R)$ or N. R is called semiperfect if R/N is semisimple artinian and if idempotents modulo N can be lifted to R.

A ring R is called right valuation if the set of right ideals of R is linearly ordered. R is called right duo if each right ideal is two-sided.

If $a \in R$, $r(a)$ will denote the right annihilator of a in R. All modules are right and unital. By a proper cyclic R-module we mean a cyclic R-module $C \ncong R$.

3. Preliminary Results.

Theorem 1 (Jain-Mohamed [2]). *If R is semiperfect such that each cyclic R-module is continuous then $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right valuation right duo rings with nil radical.*

Let R be continuous as an R-module. Then R/N is a regular ring such that the idempotents can be lifted to R (see [5]). Clearly, if each cyclic R-module is continuous then each cyclic R/A-module is continuous where A is an ideal in R. Thus by (Corollary 2.7, [2]), we obtain that R/N is semisimple artinian. Hence R is semiperfect. Thus Theorem 1 gives

Proposition 1. *Let R be a ring whose cyclic modules are continuous. Then each prime ideal is maximal and hence N is nil.*

Proposition 2. *Let R be a ring such that each proper cyclic R-module is continuous. Then R is prime or semiperfect with nil radical.*

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Proof. If R is not prime, then there exists nonzero ideals A, B in R such that $AB = 0$. Hence any prime ideal P lies above A or B. But since both R/A and R/B satisfy the hypothesis of Theorem 1, we obtain that R has only finitely many maximal ideals. But then it is obvious that R is semiperfect with nil radical.

We shall often use the basic fact for continuous modules that if $A \times B$ is a continuous module and if $\sigma\colon A \to B$ is a monomorphism then $\sigma(A)$ is a direct summand of B. Also, recall that the indecomposable continuous module is uniform.

4. Main Theorem. Let R be a semiperfect ring. Then there exists a maximal family of primitive orthogonal idempotents (e_i), $1 \leqq i \leqq n$, such that

$$1 = e_1 + e_2 + \cdots + e_n, \quad \text{and} \quad e_i R e_i$$

are local rings.

Theorem. *Let R be a semiperfect ring with nil radical. Then each proper cyclic R-module is continuous if and only if R is one of the following types:*

(a) $\oplus \sum_{\text{finite}} A_i$, *$A_i$ is simple artinian, right valuation right duo ring, or local ring with unique maximal ideal M such that $M^2 = (0)$ and $l(M) = 2$.*

(b) $\begin{pmatrix} \Delta & D \\ 0 & D \end{pmatrix}$ *where D is a division ring and Δ is a subdivision ring of D.*

Proof. Assume that each proper cyclic R-module is continuous. Let $L_r(R)$ denote the lattice of right ideals of R.

Case 1. R is local. Then for all $0 \neq I \in L_r(R)$, R/I is continuous and hence uniform. We claim that if there exist nonzero right ideals A, B of R such that $A \cap B = (0)$ then A, B are minimal right ideals and $S = A \oplus B$, where S is the right socle of R. For, if $0 \neq X \in L_r(R)$ and $X \subset A$ then R/X is uniform. But $A/X \cap B/X = (0)$ gives $A = X$. It is immediate now that $S = A \oplus B$. Let M be the unique maximal ideal of R, and let $x \in M$, $x \notin S$. Then xR must be essential right ideal, for otherwise xR will be minimal. This implies $S \subset xR$, and thus xR cannot be indecomposable. Therefore $xR = X_1 \oplus X_2$ for some nonzero $X_1, X_2 \in L_r(R)$. But then $S = X_1 \oplus X_2$, a contradiction since $x \notin S$. Hence $S = M$, which gives $M^2 = (0)$ and $l(M) = 2$.

Next, if each nonzero right ideal is essential, it follows immediately that R is right valuation. To prove R is right duo, let $aR \in L_r(R)$ and $x \in R$. Then either $xaR \subset aR$ or $aR \subset xaR$. If $xaR \subset aR$, for all $x \in R$, then aR is 2-sided. So assume $aR \subset xaR$ for some $x \in R$. In case $x \notin M$ then $xaR \cong aR$. Since xaR is continuous, we get $xaR = aR$. In case $x \in M$, consider $1 - x$ and proceed as before. Thus aR is a 2-sided ideal. Hence R is a right valuation right duo ring.

Case 2. R is not local.

We show that if for some $i, j, i \neq j, e_i R e_j \neq 0$, then every nonzero homomorphism $\sigma\colon e_j R \to e_i R$ is a monomorphism, and $e_j R e_j$ is a division ring.

Let $T = e_j R/\ker \sigma x e_i R$. T is cyclic R-module. If $T \approx R$, then $\ker \sigma = (0)$ but if $T \not\approx R$ then T is continuous which gives that the induced monomorphism

$$\sigma^*: e_j R/\ker \sigma \to e_i R$$

splits. This implies $e_j R/\ker \sigma \cong e_i R$, and hence $\ker \sigma = (0)$. Thus in each case, we get σ is a monomorphism.

To show $e_j R e_j$ is a division ring, we prove $e_j N e_j = (0)$. If possible let

$$0 \neq e_j x e_j \in e_j N e_j .$$

Let

$$\eta: e_j R \to e_j R$$

be given by $\eta(e_j y) = e_j x e_j y$. If $e_j N e_j \neq 0$, then $\eta \neq 0$. Also, since N is nil, $\ker \eta \neq (0)$. Let $0 \neq \sigma \in \hom(e_j R, e_i R)$. Then $0 \neq \sigma\eta \in \hom(e_j R, e_i R)$ and $\ker \sigma\eta \neq 0$, a contradiction. Hence $e_i N e_i = (0)$.

Consider the case $R \cong e_i R \oplus e_j R$. Let $e_i R e_j \neq (0)$. If $e_j R e_i$ is not zero then $e_i R$, $e_j R$ are subisomorphic to each other. It follows then $e_i R \cong e_j R$. Hence R is simple artinian which is of type (a). But if $e_j R e_i = (0)$, then $e_i R$, $e_j R$ are subisomorphic to each other. It follows then $e_i R \cong e_j R$. Hence R is simple artinian which is of type (a). But if $e_j R e_i = (0)$ then $e_j N = e_j N(e_i + e_j) = e_j N e_j = (0)$ implies $e_j R$ is a minimal right ideal. In this case, since $e_i R$ is uniform, for all $0 \neq e_i x e_j \in e_i R e_j$, $e_i x e_j R$ is the unique minimal right ideal in $e_i R$. But then $e_i x e_j R = e_i R e_j R$, for all $0 \neq e_i x e_j \in e_i R e_j$. This implies $e_i R e_j$ is a one-dimensional right vector space over $e_j R e_j$. It also follows that $e_i R e_j$ is a faithful left $e_i R e_i$-module. Then

$$e_i R e_i \text{ is embeddable in } \hom(e_i R e_j, e_i R e_j) \cong e_j R e_j .$$

Hence $e_i R e_i$ is a subring of a division ring $e_j R e_j$. But since N is nil, the nil subring $e_i N e_i$ of $e_i R e_i$ must be zero. Therefore $e_i R e_i$ is a division ring. We then get

$$R = \begin{pmatrix} e_i R e_i & e_i R e_j \\ 0 & e_j R e_j \end{pmatrix} \cong \begin{pmatrix} \Delta & D \\ 0 & D \end{pmatrix}$$

where D is a division ring, and Δ is a subdivision ring of D. This gives the type (b) in the theorem.

Assume now $R \not\cong e_i R \oplus e_j R$, and $e_i R e_j \neq (0)$.

Then

$$e_i R \times e_j R \cong e_i R \times \sigma(e_j R), \quad o \neq \sigma \in \hom(e_j R, e_i R e_j).$$

Since $R \not\cong e_i R \times e_j R$, $e_i R \times \sigma(e_j R)$ is continuous. But then $\sigma(e_j R) = e_i R$, and so $e_i R \cong e_j R$. Let $[e_i R] = \sum e_i R$, where the $\sum$ runs over all i for which $e_i R \cong e_i R$. Renumbering, if necessary, we may write

$$R = [e_1 R] \oplus \cdots \oplus [e_k R]$$

where $k \leqq n$. It follows from above $[e_i R]$, $1 \leqq i \leqq k$, are ideals in R. If we set $A_i = [e_i R]$, we get $R = \oplus \sum_{i=1}^{n} A_i$ as a finite direct sum of rings A_i. Each A_i is continuous as R-module (and hence A_i-module) and indeed all cyclic A_i-modules of A_i are continuous. Then A_i is simple artinian or right valuation right duo [1]. This gives type (a) in the theorem.

Finally, if all $e_i R e_j = 0 \, i \neq j$, then each $e_i R$ is an ideal in R, and R is a direct sum of local rings $e_i R$ each of whose cyclic module is continuous. Then again we obtain R is of type (a) as above.

The converse is straightforward verification and is omitted, completing the proof.

Remark 1. In view of Proposition 2 the above theorem characterizes the class of nonprime rings R each of whose proper cyclic module is continuous.

Remark 2. If R is prime we have not been able to obtain the characterization. In a special case we can show that if R is a prime local ring then R is a right valuation domain. The proof that R is right valuation is the same as the proof given earlier in the local case. Then we can show that for all $a \in R$, either $r(a) = 0$ or aR is a 2-sided ideal. This implies R has no nil elements and hence R is a domain.

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