Sonderdruck aus
Arch. Math., Vol. 48, 109–115 (1987)

0003-889 X/87/4802-0109 $ 2.90/0

Rings whose (proper) cyclic modules have cyclic π-injective hulls

By

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A module M over a ring R is called π-injective (also called quasi-continuous) if for every pair of R-submodules N_1, N_2 of M with $N_1 \cap N_2 = (0)$ each projection $\pi_i: N_1 \oplus N_2 \to N_i$, $i = 1, 2$, can be lifted to an endomorphism of M [1] and [4]. The purpose of this article is to study rings with finite uniform dimension such that every cyclic R-module is π-injective, or more generally has cyclic π-injective hull. These results generalize the earlier known results for semiperfect rings over which each cyclic R-module has injective or quasi-injective hull. In the last section we observe that characterization of semiperfect rings with nil radical over which each proper cyclic R-module is π-injective can be derived in the same manner as for the rings over which each proper cyclic R-module is continuous [2].

1. Notation and definitions. All rings have unity, and unless otherwise stated all modules are unital right modules. R is called π-hypercyclic if each cyclic R-module has a cyclic π-injective hull. It is called pcπi-ring if each proper cyclic R-module is π-injective. R is called a πc-ring if every cyclic R-module is π-injective. An R-module M is uniform if for every submodules A, B of M, $A \cap B \neq 0$. A module M_R is said to have finite uniform dimension, denoted by dim M, if M does not contain an infinite set of independent submodules. If M_R has finite uniform dimension then there exists a positive integer n such that

$$n = \inf\{m \mid m \text{ is a positive integer such that card } G \leqq m, \text{ for every independent family } G \text{ of submodules of } M\};$$

J or $J(R)$ will denote the (Jacobson) radical of R. N will denote the lower nil radical; so $J = N$ if J is nil. The right singular ideal of R, denoted by $Z(R)$, is the two sided ideal $\{r \in R \mid Ir = 0$, for some essential right ideal I of $R\}$.

R is called a local ring if it has a unique maximal right ideal $M (= J(R))$. R is called right valuation if right ideals of R are linearly ordered.

$A_R \subset' B_R$ shall mean A_R is essential (large) in B_R. The injective hull of A_R is denoted by $E_R(A)$ (or simply by $E(A)$). The quasi-injective hull of A is denoted by q. inj. hull (A). Let $K = \mathrm{Hom}_R(E(A), E(A))$. Then the q. inj. hull $(A) = KA$ [5]. The π-injective hull of A, denoted by $\pi(A)$, is the minimal π-injective (or quasi-continuous) extension of A. The π-injective hull of A is $\pi(A) = UM$, where U is the subring of K generated by the idempotents of K. Clearly, $\pi(A) \subset'$ q. inj. hull $(A) \subset' E(A)$.

2. Preliminary results.

2.1. Lemma. *Let M_R be π-injective. If $E(M) = \oplus \sum_{i=1}^{\bullet} A_i$ is a direct sum of submodules A_i, then $M = \oplus \sum_{i=1}^{\bullet} (M \cap A_i)$.*

Proof. See ([3], Theorem 1.1).

2.2. Lemma. *Let R be a ring with finite uniform dimension. If the π-injective hull of R is cyclic, then R is π-injective.*

Proof. Suppose $\pi(R) \cong R/I$ for some right ideal I of R. There exists B_R such that $R \cong B/I \subset' R/I$. Thus $B \cong I \oplus K$ for some $K_R \cong R_R$, and $B \subset' R$. Now $\dim R = \dim B = \dim (I \oplus K) = \dim I + \dim R$ which forces I to be 0. That is $\pi(R) = R$. □

2.3. Lemma. *Let R be π-hypercyclic ring. Then a ring homomorphic image of R is also π-hypercyclic.*

Proof. Let A be a two sided ideal of R. Let $\bar{R} = R/A$ and let $\bar{R}/\bar{I}$ be a cyclic $\bar{R}$-module, where $\bar{I} = I/A$. By hypothesis there exists K_R such that R/K is the π-injective hull of R/I as an R-module. $R/K \cong U(R/I)$, where U is the subring of $\mathrm{Hom}_R(E_R(R/I), E_R(R/I))$ generated by the idempotents. Since A annihilates R/I, it follows that A annihilates R/K. Thus R/K is an $\bar{R}$-module. Clearly, $\bar{R}/\bar{K}$ ($\cong R/K$) is the π-injective hull of $\bar{R}/\bar{I}$ ($\cong R/I$) as an $\bar{R}$-module. This proves that $\bar{R}$ is π-hypercyclic. □

Remark. A ring homomorphic image of a πc-ring is a πc-ring.

2.4. Lemma. *Let R be a ring with finite uniform dimension. Then R is πc-ring (π-hypercyclic ring) iff $R = e_1 R \oplus \cdots \oplus e_m R$, where $e_i R$ is an indecomposable πc-ring (π-hypercyclic ring),* $1 \leqq i \leqq m \leqq \dim R$.

Proof. Follows from the fact that R cannot have infinite set of orthogonal idempotents, and that ring homomorphic image of a πc-ring (or π-hypercyclic ring) is again a πc-ring (or π-hypercyclic ring). □

2.5. Lemma. *A π-hypercyclic ring (or πc-ring) R is local iff R is right valuation.*

Proof. Let R be π-hypercyclic. Then by Lemma 2.2, R_R is uniform. Let A, B be non zero right ideals of R. Then $A \cap B \neq 0$. Suppose $\frac{A}{A \cap B} \neq 0$; $\frac{B}{A \cap B} \neq 0$. $\frac{A}{A \cap B} \oplus \frac{B}{A \cap B} \subseteqq \frac{R}{A \cap B} \subseteqq \pi\left(\frac{R}{A \cap B}\right) = R/K$, say. Since R is local, R/K is an indecomposable π-injective R-module, and hence R/K is uniform. Thus either $\frac{A}{A \cap B} = 0$ or $\frac{B}{A \cap B} = 0$, that is $A \subseteqq B$ or $B \subseteqq A$. It follows that R is right valuation. The converse is trivial. □

3. πc-rings with finite uniform dimension. In view of Lemma 2.4 it suffices to consider indecomposable πc-rings (or π-hypercyclic rings). Thus in this and in the following section we restrict ourselves to indecomposable rings.

3.1. Theorem. *Let R be an indecomposable πc-ring. Then R has finite uniform dimension iff R is simple artinian or it is a right uniform ring.*

Proof. Let $\dim R = m$. Then $E(R) = E_1 \oplus \cdots \oplus E_m$, where E_i is an indecomposable injective R-module, $1 \leqq i \leqq m$. By Lemma 2.1 we have, $R = E_1 \cap R \oplus \cdots \oplus E_m \cap R$, a finite direct sum of uniform right ideals. If R is not uniform, then there exist a set of non-zero orthogonal idempotents $e_i, 1 \leqq i \leqq m, m > 1$ such that $R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_m R$. But then $e_i R$, $1 \leqq i \leqq m$, are isomorphic minimal right ideals of R ([1], Lemma 2.3). Thus R is simple artinian. The converse is obvious. □

The next theorem gives a sufficient condition for a πc-ring to have a finite uniform dimension.

3.2. Theorem. *Let R be a πc-ring with polynomial identity. Then R has finite uniform dimension.*

Proof. Let N be the lower nil radical of R. Then R/N is semiprime with polynomial identity. Therefore the right singular ideal $Z(R/N)$ of the ring R/N is zero. Since R/N is also a πc-ring, it follows that $R/N = K \oplus L$, where K is semisimple artinian, L is a finite direct sum of right Ore domains ([1], Theorem 2.10). Thus R/N has finite uniform dimension say n. We show $\dim R = n$. If possible, let $A_1 \oplus \cdots \oplus A_{n+1} \subset' R$, $A_i \neq 0$ for $i = 1, \ldots, n$. Then $E(R) = E(A_1) \oplus \cdots \oplus E(A_{n+1})$, and so by Lemma 2.1, $R = E(A_1) \cap R \oplus \cdots \oplus E(A_{n+1}) \cap R$. This implies that there exist nonzero orthogonal idempotents $e_i \in R$, and hence $\bar{0} \neq \bar{e}_i \in \bar{R}$, $1 \leqq i \leqq n + 1$, a contradiction. □

4. π-hypercyclic rings with finite uniform dimension. In this section we consider rings over which each cyclic module has a cyclic π-injective hull (π-hypercyclic rings). It is easy to see that if R is a commutative π-hypercyclic ring then R is a πc-ring. Theorem 4.3 characterizes indecomposable π-hypercyclic rings with finite uniform dimension greater than one. We begin with an important lemma which plays a key role in the proof of this characterization.

4.1. Lemma. *Let A be essential in an injective module B. Then $\pi(A \times B) \cong E(A) \times B$.*

Proof. Let $\pi = \pi(A \times B)$. By Lemma 2.1, $\pi = (\pi \cap E(A)) \times B$. Since $A \subset' B$, $E(\pi \cap E(A)) \cong B = E(B)$. Thus it follows by ([1], Proposition 1.11) that $\pi \cap E(A) \cong B$. Therefore, π is injective, and hence $\pi = E(A) \times B$. □

The following is well known.

4.2. Lemma. *Let R/J be artinian and let I be a right ideal of R. If $R/I = \oplus \sum_{i=1}^{k} N_i$, then $k \leqq$ composition length of R/J.*

4.3. Theorem. *Let R be a π-hypercyclic indecomposable ring with finite uniform dimension other than 1. Then $R = A_n$, where A is a right valuation ring, $n > 1$ iff R is self injective.*

Proof. Let R be self-injective. Thus there exist primitive orthogonal idempotents e_i, $1 \leqq i \leqq n$, such that $R = e_1 R \oplus \cdots \oplus e_n R$. Let α: $e_1 R \to e_2 R$ be a nonzero R-homomorphism. By Lemma 4.1 $R/I = \pi\left(\frac{R}{\ker \alpha}\right) \cong e_2 R \times e_2 R \times \cdots \times e_n R$, for some right ideal I of R. Thus R/I is projective and hence $R \cong I \oplus K$ where $K \cong e_2 R \times e_2 R \times \cdots \times e_n R$. Therefore, $e_1 R \times e_2 R \times \cdots \times e_n R \cong I \times e_2 R \times e_2 R \times \cdots \times e_n R$. Since R is self-injective, Azumaya Diagrams give $e_1 R \cong I \times e_2 R$ which forces I to be zero since $e_1 R$ is indecomposable. Thus, $e_1 R \cong e_2 R$. Let $[e_k R] = \Sigma e_j R$ such that $e_j R \cong e_k R$. Since R is indecomposable $R = [e_1 R]$. Therefore, R is a matrix ring over a local ring $A \cong eRe$, $e = e_1$. It remains to show that eRe is a right valuation ring. We first show that for any right ideal $I \subset eR$, $\frac{eR}{I}$ is uniform. Since R is π-hypercyclic, $\pi(R/I) = R/K$. Therefore, by Lemma 4.2, $\dim \pi(R/I) \leqq n$. Further, $E(R/I) = E(eR/I) \oplus (1 - e)R$ and an application of Lemma 2.1 gives $\dim \pi\left(\frac{eR}{I}\right) = \dim E\left(\frac{eR}{I}\right) = 1$, proving $\frac{eR}{I}$ is uniform. Now, it follows immediately that the submodules of eR are linearly ordered. We proceed to show eRe is a valuation ring: Let A, B be right ideals of eRe. Then, $AeR \subset (eR)^2$ and $BeR \subset (eR)^2$. Since the submodules of eR are linearly ordered, $AeR \subset BeR$ or $BeR \subset AeR$, and so $AeRe \subset BeRe$ or $BeRe \subset AeRe$, i.e. $A \subset B$ or $B \subset A$. The converse follows from the fact that an $n \times n$ matrix ring A_n, $n > 1$, is self π-injective iff it is self-injective.

5. pcπI-rings. In this section we obtain a characterization of semiperfect rings with nil radical such that each proper cyclic module has cyclic π-injective hull. It follows then that each proper cyclic R-module is π-injective iff each proper cyclic R-module is continuous where R is semiperfect nonlocal ring with nil radical. The technique for obtaining the stated characterization is similar to the one for rings over which proper cyclic modules are continuous. Hence we shall often refer the reader to the proof in [2] and shall merely prove the lemmas needed to replace an argument so as to fit in the more general situation. Recall that we call a ring R to be a pcπi-ring if each proper cyclic R-module is π-injective.

5.1. Lemma. *Let A and B be R-modules such that A is embeddable in B. If $A \times B$ is π-injective then the exact sequence $0 \to A \to B$ splits.*

Proof. See ([1], Cor. 1.13). □

Repeated application of Lemma 5.1 yields

5.2. Theorem. *Let $R = \oplus \sum_{i=1}^{n} e_i R$, $n > 2$, be a semiperfect ring, where e_i, $1 \leqq i \leqq n$, is a family of primitive orthogonal idempotents. Then R is a pcπi-ring iff $R = \oplus \sum_{\text{finite}} R_i$, where R_i is a simple artinian ring or a right valuation ring.*

Proof. By using Lemma 5.1, the proof is similar to the proof of the corresponding result in [2]. We may note that we do not need to assume that the radical is nil. □

If A, B are continuous R-modules such that A is embeddable in B and B is embeddable in A, then it is well known that $A \cong B$. However, this is not true for π-injective modules. This necessitates the separate treatment of the semiperfect rings $R = e_1 R \oplus e_2 R$, where e_1, e_2 are primitive orthogonal idempotents.

5.3. Theorem. *Let $R = e_1 R \oplus e_2 R$ be a semiperfect ring with nil radical where e_1, e_2 are primitive orthogonal idempotents. Suppose $e_1 Re_2 \neq 0$ and $e_2 Re_1 \neq 0$. Then R is a pcπi-ring iff R is simple artinian.*

Proof. Let R be a pcπi-ring. By hypothesis and Lemma 5.1, $e_1 R$ is embeddable in $e_2 R$ and vice versa. Thus either $e_1 R \cong e_2 R$, or $e_1 R \hookrightarrow e_2 N$ and $e_2 R \hookrightarrow e_1 N$. The latter implies $R \hookrightarrow N$ and so $R \cong nR$, $n \in N$ where ann $n = 0$, a contradiction since N is nil. Thus $e_1 R \cong e_2 R$. We now show $e_1 Ne_1 = 0$. Let $(e_1 x e_1)^{k-1} \neq 0$, $(e_1 x e_1)^k = 0$, $x \in N$. Then consider the mapping $f: e_1 R \to e_1 R$ given by $f(e_1 y) = e_1 x e_1 y$. Clearly, $0 \neq (e_1 x e_1)^{k-1} \in \ker f$. This implies for each $0 \neq g \in \mathrm{Hom}_R(e_1 R, e_2 R)$, $\ker gf \neq 0$, a contradiction to the fact that any nonzero homomorphism from $e_1 R$ to $e_2 R$ must be a monomorphism. Hence, $e_1 Ne_1 = 0$. This proves R is simple artinian. □

We continue our consideration of a semiperfect ring $R = e_1 R \oplus e_2 R$ but under the hypothesis that one of the additive groups $e_1 Re_2$, $e_2 Re_1$ is not zero and the other is zero. The following is a key lemma towards the characterization of such pcπi-rings.

5.4. Lemma. *Let $R = e_1 R \oplus e_2 R$ be a semiperfect pcπi-ring with nil radical such that $e_1 Re_2 \neq 0$, $e_2 Re_1 = 0$. Then $e_2 R$ is a minimal right ideal and $\mathrm{ann}_{e_1 Re_1} e_1 Re_2 = e_1 Ne_1$.*

Proof. Since any nonzero homomorphism from $e_2 R$ to $e_1 R$ must be a monomorphism, it follows that $(e_1 x e_2)(e_2 y) = 0$ implies either $e_1 x e_2 = 0$ or $e_2 y = 0$. Let $(e_2 y)^2 = 0$, $e_2 y \neq 0$. Choose $0 \neq e_1 x e_2 \in e_1 Re_2$. Then $(e_1 x e_2 y e_2) e_2 y = 0 \Rightarrow e_1 x e_2 y e_2 = 0 \Rightarrow e_2 y e_2 = 0 \Rightarrow e_2 y = 0$, since $e_2 Re_1 = 0$. Hence $e_2 N = 0$, and so $e_2 R$ is a minimal right ideal of R. This implies for each nonzero $e_1 x e_2 \in e_1 Re_2$, $e_1 x e_2 R$ is the unique minimal right ideal ($\cong e_2 R$) in each nonzero right ideal of $e_1 R$. Next let $e_1 x e_1 \in e_1 Ne_1$ such that $(e_1 x e_1)^k = 0$, $(e_1 x e_1)^{k-1} \neq 0$. Then $0 \neq (e_1 x e_1)^{k-1} \in \mathrm{ann}_{e_1 R}(e_1 x e_1)$ and so $e_1 Re_2 R \subset \mathrm{ann}_{e_1 R}(e_1 x e_1)$. This implies $e_1 x e_1 Re_2 R = 0$ for each $x \in N$, proving the lemma.

5.5. Lemma. *Under the hypothesis of Lemma 5.4.*

$$A = \begin{bmatrix} e_1 Ne_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{is an ideal in} \quad S = \begin{bmatrix} e_1 Re_1 & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix}$$

and

$$\frac{S}{A} \cong \begin{bmatrix} (e_1 Re_1)/(e_1 Ne_1) & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix}$$

is not π-injective as S or S/A-module.

Proof. It follows from Lemma 5.4 that $e_1 Re_2$ is a 1-dimensional right vector space over $e_2 Re_2$.

Thus we may write

$$\begin{bmatrix} (e_1 Re_1)/(e_1 Ne_1) & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix} = \begin{bmatrix} \Delta & D \\ 0 & D \end{bmatrix}$$

where Δ, D are division rings. Clearly S/A is canonically isomorphic to $\begin{bmatrix} \Delta & D \\ 0 & D \end{bmatrix}$. Also

$$\begin{bmatrix} \Delta & D \\ 0 & D \end{bmatrix} = \begin{bmatrix} \Delta & D \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

as S/A-modules. Since $\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ is embeddable in $\begin{bmatrix} \Delta & D \\ 0 & 0 \end{bmatrix}$, it follows by Lemma 5.1 that if S/A is π-injective then the embedding must be isomorphism, which is clearly not possible. This completes the proof. □

5.5. Theorem. *Let $R = e_1 R \oplus e_2 R$ be a semiperfect ring with nil radical such that e_1, e_2 are primitive orthogonal idempotents, $e_1 Re_2 \neq 0$, $e_2 Re_1 = 0$. Then R is a pcπi-ring iff $R \cong \begin{bmatrix} \Delta & D \\ 0 & D \end{bmatrix}$ where D is a division ring and Δ is a subdivision ring of D.*

Proof. Since each proper ring homomorphic image of R is a πc-ring it follows by Lemmas 5.4 and 5.5 that $\mathrm{ann}_{e_1 Re_1} e_1 Re_2 = 0$. Hence $e_1 Re_2$ is a faithful left $e_1 Re_1$-module and since it is also a right $e_2 Re_2$ 1-dimensional vector space, it follows that $\Delta = e_1 Re_1$ is a subdivision ring of $D = e_2 Re_2$. Thus the "only if" part of the theorem is completed. The "if" part follows by verifying that R/A is π-injective for each nonzero right ideal A. The details are straightforward. □

To complete our discussion we need to give characterization of local pcπi-rings. This is contained in the following theorem whose proof is similar to the proof of local rings over which every proper cyclic module is continuous. However, we note that the characterization in the case of local pcπi-rings is somewhat different.

5.7. Theorem. *Let R be a local ring with unique maximal ideal N. Then R is pcπi-ring iff R is either a right valuation ring or $N^2 = (0)$, and the composition length of N is 2.*

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Journal of the Indian Math. Soc.
42 (1978) 197–202

RINGS WHOSE CYCLIC MODULES ARE CONTINUOUS

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[Received July 12, 1976: revised January 17, 1977]

RINGS WHOSE CYCLIC modules are quasi-injective were studied by Ahsan [1], and fully characterized by Koehler [2] as semiperfect rings which are finite direct sum of rings each of which is simple artinian or rank zero maximal valuation duo rings. Following Utumi [7] Mohamed and Bouhy [4] introduced the notion of a continuous module as a generalization of quasi-injective module. They also characterized a ring whose finitely generated modules are continuous as semisimple artinian. In this paper, we study the class of rings whose cyclic modules are continuous. Such rings are called right *cc*-rings. Our main theorem is: A semiperfect ring R is a right *cc*-ring if any only if R is a finite product of rings which are either simple artinian or right valuation right duo with nil radical.

1. Definitions, Notations and preliminaries. All rings considered have unities and all modules are unital right modules. A module M is indecomposable if 0 and M are the only direct summands of M. A module M is uniform if every two nonzero submodules of M have non-trivial intersection. An idempotent e of a ring R is indecomposable if the right R-module eR is indecomposable. A ring R is local if it has exactly one maximal right ideal. Rad R will stand for the Jacobson radical of a ring R. A ring R is semiperfect if and only if $R/\text{Rad } R$ is semisimple artinian and idempotents modulo Rad R can be lifted. R is a right valuation ring if for every two right ideals A and B of R either $A \subset B$ or $B \subset A$. R is a right duo ring if every right ideal of R is a two-sided ideal. If X is a subset of a ring R then $X^{\perp}$ will denote the right annihilator of X in R. A module M is embedded in a module N (notation $M \subsetneq N$) if there is a monomorphism of M into N.