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π-INJECTIVE MODULES AND RINGS WHOSE CYCLICS ARE π-INJECTIVE

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A module M is said to be π-injective if for every pair M_1, M_2 of R-submodules of M with $M_1 \cap M_2 = (0)$, each projection $\pi_i : M_1 \oplus M_2 \to M_i$, i = 1,2, can be lifted to an endomorphism of M. Clearly, each injective (or even quasi-injective) module is π-injective. But if R is a right Öre-domain which is not a division ring then R is not injective but it is trivially π-injective; indeed a module M is uniform if and only if M is π-injective and indecomposable. In section 1 various results regarding π-injective modules are obtained. As an application it follows that π-injective modules over a right noetherian ring is a direct sum of indecomposable π-injective modules. Also if A is self π-injective and G is a finite group then the group ring R = AG is self π-injective. Conversely, if R = AG is self π-injective then A is self π-injective but G need not be finite. In section 2 rings whose cyclic modules are π-injective are considered. It is shown that if R is a self-injective ring (or if R is a semiperfect ring) then each cyclic R-module is π-injective if and only if R is a direct sum of semisimple artinian ring and a finite direct sum of right valuation rings. Also a ring with zero singular ideal each of whose cyclic module is π-injective is shown to be a direct sum of semisimple artinian ring and a finite direct sum of right Öre-domains. The last two results are generalizations of a wellknown theorem of Osofsky for rings whose cyclics are injective.

§1. π-Injective Modules

Throughout all modules are right and unital. $\hat{M}$ will denote the injective hull of M and hom(M,M) will denote the endomorphism ring of R-homomorphisms of M. For any submodule N of M, N^c will denote a complement of N in M. A submodule N of M is called closed if N has no proper essential extension in M. $N \overset{\cdot}{\subset} M$ shall denote that N is an essential submodule of M.

Theorem 1.1. For any R-module M the following are equivalent:

(a) M is π-injective.

(b) For every idempotent e in $\text{hom}(\hat{M},\hat{M})$, $eM \subset M$.

(c) If $\hat{M} = N_1 \oplus N_2$ then $M = (N_1 \cap M) \oplus (N_2 \cap M)$.

(d) If $\hat{M} = \oplus \sum_{i \in \Lambda} N_i$ then $M = \oplus \sum_{i \in \Lambda} (N_i \cap M)$ for any index set Λ.

Proof. (a) $\Rightarrow$ (b). Let e be an idempotent in $\text{hom}(\hat{M},\hat{M})$. Set $M_1 = M \cap e\hat{M}$ and $M_2 = M \cap (1-e)\hat{M}$. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the projection. Since M is π-injective, π_1 can be lifted to an endomorphism f in hom(M,M). Also f can be lifted further to an endomorphism g in hom($\hat{M},\hat{M}$). Suppose (e-g)M ≠ (0), then M ∩ (e-g)M ≠ (0). Let (e-g)x = y for some nonzero x,y ∈ M. Clearly, ex ∈ M_1 and (1-e)x ∈ M_2. Therefore, x = ex + (1-e)x ∈ $M_1 \oplus M_2$. This yields (e-g)x = 0, contrary to the assumption that y is nonzero. Thus (e-g)M = (0) and hence eM = gM = fM ⊂ M.

(b) ⇒ (c). Assume $\hat{M} = N_1 \oplus N_2$. Let $\pi_i : N_1 \oplus N_2 \to N_i$, i = 1,2, be projections. Then π_i are idempotents in hom($\hat{M},\hat{M}$). Thus from (b), $\pi_1 M \subset M$ and $\pi_2 M \subset M$. Hence $M = (\pi_1+\pi_2)M \subset \pi_1 M + \pi_2 M \subset (N_1 \cap M) + (N_2 \cap M)$, proving $M = (N_1 \cap M) \oplus (N_2 \cap M)$.

(c) ⇒ (d). Straightforward verification.

(d) ⇒ (a). Let M_1, M_2 be submodules of M such that $M_1 \cap M_2 = (0)$. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the projection. By Zorn's lemma there exists M_3, a submodule of M, such that $(M_1 \oplus M_2) \oplus M_3 \overset{!}{\subset} M$. Then $\hat{M} = \hat{M}_1 \oplus \hat{M}_2 \oplus \hat{M}_3$. Thus by (d), $M = (\hat{M}_1 \cap M) \oplus (\hat{M}_2 \cap M) \oplus (\hat{M}_3 \cap M)$. Define e ∈ hom(M,M) such that e(x) = x for all $x \in (\hat{M}_1 \cap M)$ and e(y) = 0 for all $y \in (\hat{M}_2 \cap M) \oplus (\hat{M}_3 \cap M)$. Clearly, e is an extension of π_1. Similarly, we can extend the projection $\pi_2 : M_1 \oplus M_2 \to M_2$ to an endomorphism of M.

The next theorem shows the existance and uniqueness (up to isomorphism) of a minimal π-injective essential extension of a module.

Theorem 1.2. Let M be any R-module, $K = \hom(\hat{M},\hat{M})$ and V be the subring of K generated by all the idempotents in K. Then M is π-injective iff M is (V,R)-submodule of $\hat{M}$.

Proof. Follows from theorem 1.1.

Corollary 1.3. Each R-module M has a unique minimal π-injective essential extension in $\hat{M}$ (to be called π-injective hull of M in $\hat{M}$). Also if $M_1 \cong M_2$ then their π-injective hulls are also isomorphic.

Henceforth we shall denote by M^{π} the π-injective hull of M.

Proposition 1.4. Let M be an R-module and U be the subring generated by all the idempotents in $\hom(M^{\pi},M^{\pi})$. Then $UM = M^{\pi}$.

Proof. Clearly, $UM \subset M^{\pi}$. Since for all idempotents e in $\hom(\hat{M},\hat{M})$, $eM \subset M^{\pi}$, it follows that UM is π-injective and hence $UM = M^{\pi}$.

Let M be an R-module with the singular submodule $Z(M) = (0)$. Then each f in $\hom(M,M)$ can be uniquely extended to g in $\hom(\hat{M},\hat{M})$. This gives an embedding of the ring $\hom(M,M)$ into the ring $\hom(\hat{M},\hat{M})$. In the proposition which follows we shall identify the ring $S = \hom(M,M)$ with its copy in the ring $K = \hom(\hat{M},\hat{M})$.

Proposition 1.5. Let M be an R-module with $Z(M) = (0)$. Then M is π-injective iff $V = W$, where V and W are the subrings of K and of S generated by all the idempotents in $K = \hom(\hat{M},\hat{M})$ and $S = \hom(M,M)$ respectively.

Proof. Apply theorem 1.1.

Proposition 1.6. A module M is uniform iff M is π-injective and indecomposable.

Proof. Let N be a nonzero submodule of M and N^c be a complement of N in M. We can write $\hat{M} = \hat{N} \oplus \hat{N}^c$. By theorem 1.1, $M = (M \cap \hat{N}) \oplus (M \cap \hat{N}^c)$. Since M is indecomposable we get $N^c = (0)$. Hence M is uniform. The converse is obvious.

Proposition 1.7. A direct summand of π-injective module is π-injective.

Proof. Obvious.

Proposition 1.8. Let M be a π-injective R-module. Then every closed submodule N of M is a direct summand of M. Also the sum of two closed submodules of M is closed.

Proof. Let N be a closed submodule and N^c a complement of N in M. By theorem 1.1, $M = (M \cap \hat{N}) \oplus (M \cap \hat{N}^c)$. Since N is closed in M, $N = M \cap \hat{N}$ and hence $N^c = M \cap \hat{N}^c$. Thus $M = N \oplus N^c$. The proof of the latter part is omitted.

Corollary 1.9. Let M be π-injective. If N is closed then N is π-injective.

We now give an analogue of Matlis theorem.

Theorem 1.10. Let R be a right noetherian ring and M be a π-injective module. Then M is a direct sum of indecomposable π-injective R-modules.

Proof. Since R is noetherian, by Matlis theorem, $\hat{M} = \oplus \sum_{\alpha\in\Lambda} N_\alpha$, where each N_α is indecomposable. Put $M_\alpha = M \cap N_\alpha$. Further, since M is π-injective $M = \oplus \sum_{\alpha\in\Lambda} (M \cap N_\alpha)$ by theorem 1.1. Clearly, each M_α is π-injective and indecomposable.

Proposition 1.11. Let M_1, M_2 be R-modules such that $M_1 \oplus M_2$ is π-injective and $\hat{M}_1 \cong \hat{M}_2$. Then $M_1 \cong M_2$.

Proof. Let $g:\hat{M}_1 \to \hat{M}_2$ be the given isomorphism. Consider an idempotent $\begin{pmatrix} 1 & 0 \\ g & 0 \end{pmatrix} \in \begin{pmatrix} \hom(\hat{M}_1,\hat{M}_1) & \hom(\hat{M}_2,\hat{M}_1) \\ \hom(\hat{M}_1,\hat{M}_2) & \hom(\hat{M}_2,\hat{M}_2) \end{pmatrix} \cong$ $\hom(\hat{M}_1 \oplus \hat{M}_2, \hat{M}_1 \oplus \hat{M}_2)$. Since $M_1 \oplus M_2$ is π-injective $\begin{pmatrix} 1 & 0 \\ g & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in M_1 \oplus M_2$ for all $m_1 \in M_1$, $m_2 \in M_2$. This implies $gm_1 \in M_2$ and hence $gM_1 \subset M_2$. Similarly, we can prove $g^{-1}M_2 \subset M_1$. Thus $gM_1 = M_2$, proving $M_1 \cong M_2$.

Note. The converse is not true. For example if M is π-injective which is not quasi-injective then $M \times M$ cannot be π-injective as follows from the proposition proved below.

Proposition 1.12. Let N,M be R-modules such that $N \times M$ is π-injective. Then N is M-injective and M is N-injective.

Proof. Let M_1 be a submodule of M and $f:M_1 \to N$ be an R-homomorphism. Then f can be lifted to $g:\hat{M} \to \hat{N}$. Consider an idempotent $\begin{pmatrix} 0 & g \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} \hom(\hat{N},\hat{N}) & \hom(\hat{M},\hat{N}) \\ \hom(\hat{N},\hat{M}) & \hom(\hat{M},\hat{M}) \end{pmatrix} \cong \hom(\hat{N}\times\hat{M}, \hat{N}\times\hat{M})$. Since $N \times M$ is π-injective, $\begin{pmatrix} 0 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix} \in N \times M$ for all $n \in N$, $m \in M$. This implies $gM \subset N$, proving that N is M-injective. Similarly, M is N-injective.

Corollary 1.13. If $N \times M$ is π-injective and $0 \to N \to M$ is an exact sequence then the sequence splits, and N is quasi-injective. In particular, if $M \times M$ is π-injective then M is quasi-injective.

We shall call a ring R to be self π-injective if R as an R-module is π-injective.

Theorem 1.14. Let $S = R_n$, $n > 1$ be the $n \times n$ matrix ring over a ring R. Then the following are equivalent:

(a) S is self π-injective.

(b) S is self-injective.

Proof. (a) ⇒ (b). Note $S = e_{11}S \oplus \dots \oplus e_{nn}S$, e_{ii} is a matrix with all entries 0 except at the (i,i)th place where the entry is 1. Also $e_{11}S \cong \dots \cong e_{nn}S$. Thus by corollary 1.13, each $e_{ii}S$ is quasi-injective as S-module and hence S is self-injective.

(b) ⇒ (a). Obvious.

Theorem 1.15. If R is self π-injective with $Z(R) = (0)$ then $R = A \oplus B$ where A and B are ideals. The ring A is regular self-injective such that (i) each of its nonzero ideal contains a nonzero nilpotent element and (ii) it is generated by idempotents. The ring B is such that $\hat{B}$ is a strongly regular ring.

Proof. Since $Z(R) = (0)$, $\hat{R}$ is a regular ring. Hence from Utumi ([5], Thm. 3.2), $\hat{R} = A \oplus C$, where every nonzero ideal of A contains a nonzero nilpotent element, A is generated by idempotents and C is strongly regular. Since R is

π-injective $R = (A \cap R) \oplus (C \cap R)$. But from proposition 1.5, $A \subset R$. Thus $R = A \oplus B$, where $B = C \cap R$ and $\hat{B} = C$, proving the theorem.

As a consequence of the above theorem we have the following interesting result.

Theorem 1.16. Let R be a prime ring with $Z(R) = (0)$. Then R is self π-injective iff R is either self-injective or right Öre-domain.

Proof. Observe that a prime self π-injective ring with no nilpotent elements is a right Öre-domain.

We now consider self π-injective group rings. It is wellknown that a group ring $R = AG$ is self-injective iff A is self-injective ring and G is a finite group. However, the finiteness of the group G is not necessary for $R = AG$ to be self π-injective. For example, if A is a field and G is an infinite cyclic group then AG is a commutative domain and hence self π-injective. More generally, if A is a right Öre-domain and G is a torsion-free locally nilpotent group then by Smith ([4], Thm. 3.4), $R = AG$ is right Öre-domain and hence self π-injective. The following theorem is an analogue of the theorem for self-injective rings.

Theorem 1.17. If A is a self π-injective ring and G is a finite group then $R = AG$ is self π-injective. Conversely, if $R = AG$ is self π-injective then A is self π-injective.

The proof follows on the same lines as that of Connell [1]. Before we sketch the proof of the above theorem we prove the following.

Proposition 1.18. Let $R = AG$ be a group ring. If M is any A-module then $H = \hom_A(R,M)$ can be made into an R-module in a natural way. Further, if M is π-injective as A-module then H is π-injective as R-module.

Proof. Let $\alpha \in H$, $r \in R$. Define $\alpha r = \alpha[r]$, where $\alpha[r](r') = \alpha(rr')$ for all $r' \in R$. Then H is an R-module. Let H_1, H_2 be two R-submodules of H such that $H_1 \cap H_2 = (0)$ and $\pi_1 : H_1 \oplus H_2 \to H_1$ be the projection. Let $M_i = H_i R = \sum_{h \in H_i} hR$, $i = 1,2$. Then M_1 and M_2 are A-submodules of M. We claim $M_1 \cap M_2 = (0)$. Let $x \in M_1 \cap M_2$. Then $x = f_1 a_1 + \ldots + f_n a_n = g_1 b_1 + \ldots + g_k b_k$ where $f_i \in H_1$, $g_i \in H_2$, a_i, $b_i \in R$. If $h \in H$, $r \in R$, define $\phi_{hr} : R \to M$ by $\phi_{hr}(s) = h(rs)$, $s \in R$. Then $\phi_{hr} \in H$. Clearly, $\phi_{f_1 a_1} + \ldots + \phi_{f_n a_n} = \phi_{g_1 b_1} + \ldots + \phi_{g_k b_k} = h$ (say). Then $h \in H_1 \cap H_2 = (0)$. Thus $h = 0$, giving $f_1 a_1 + \ldots + f_n a_n = g_1 b_1 + \ldots + g_k b_k = 0$. Hence $M_1 \cap M_2 = (0)$. Since M is π-injective as A-module there exists $\theta : M \to M$ such that $\theta(m) = m$, $m \in M_1$ and $\theta(m) = 0$, $m \in M_2$. Define $\psi : H \to H$ by $\psi(h) = \psi_{\theta h}$, $h \in H$ where $\psi_{\theta h}(r) = \theta(hr)$, $r \in R$. It is easy to check that ψ is an extension of π_1, proving H is π-injective as R-module.

Proof of the theorem 1.17. (if part) Put $M = R$ in the proposition 1.18. Then $R \cong \hom_A(R,R) = H$ is π-injective as R-module.

Conversely, let J_1 and J_2 be right ideals in A such that $J_1 \cap J_2 = (0)$. Let $\pi_1 : J_1 \oplus J_2 \to J_1$ be the projection. Cleary, $J_1R \cap J_2R = (0)$. Since R is self π-injective there exists $\Phi : R \to R$ such that $\Phi(r) = r$, $r \in J_1R$ and $\Phi(r) = 0$, $r \in J_2R$. Define $t : A \to A$ by $t(a) = \Phi(a.1)(1)$, i.e. $t(a)$ is the trace of $\Phi(a.1)$. Then $t \in \hom_A(A,A)$ and t is an extension of π_1. Thus A is self π-injective.

It is a natural question to ask if A is self π-injective then what conditions on the group G besides finiteness shall make the group ring R = AG self π-injective. In particular, if G is an infinite cyclic group and A is a prime self π-injective ring, is it true that R = AG is self π-injective?

§2. Rings Whose Cyclics are π-Injective

A ring R is called a right πc-ring if every cyclic R-module is π-injective.

Lemma 2.1. Let R be a ring. Then R is a right πc-ring iff R/I is a right πc-ring for each ideal I of R.

Proof. Straightforward.

Lemma 2.2. Let $R = R_1 \oplus \ldots \oplus R_n$ where R_i, $i = 1, \ldots, n$, are rings. Then R is a right πc-ring iff each R_i, $i = 1, \ldots, n$, is a right πc-ring.

Proof. Obvious.

Lemma 2.3. Let R be a right πc-ring. If e and f are orthogonal indecomposable idempotents of R such that $eRf \neq (0)$, then eR and fR are isomorphic minimal right ideals of R.

Proof. Choose $a \in R$ such that $eaf \neq 0$. Let r(ea) denotes the right annihilator of ea in R. Since $eafR \times eR \cong (e+f)R/(r(ea) \cap fR)$, $eafR \times eR$ is π-injective and thus by corollary 1.13, $eafR = eR$ because eR is indecomposable. Therefore, eafR is projective. Since fR is also indecomposable, $eafR \cong fR$, proving $eR \cong fR$. Now we shall prove that eR is minimal. Let $0 \neq eb \in eR$. If $eb(1-e) \neq 0$ then as before we get $eR = eb(1-e)R$. Thus $eR = ebR$. If $eb(1-e) = 0$ then $ebe \neq 0$. Since $ebeR \oplus fR = (ebe + f)R$, $ebeR \oplus fR$ is π-injective. Also $eR \cong fR$ implies $ebeR \oplus fR \cong ebeR \times eR$. Thus $ebeR \times eR$ is π-injective. Again by corollary 1.13, $ebeR = eR$, proving $eR = ebR$. Hence eR is a minimal right ideal.

A ring R is a right valuation ring if for every pair of elements x, y in R, either $xR \subset yR$ or $yR \subset xR$.

Theorem 2.4. If R is semiperfect then R is a right πc-ring iff $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right valuation rings.

Proof. Since R is semiperfect, $R = e_1R \oplus \dots \oplus e_nR$, where $\{e_1,\dots,e_n\}$ is a set of orthogonal indecomposable idempotents. Set $[e_tR] = \Sigma e_iR$, $e_iR \cong e_tR$. Renumbering if necessary, we may write $R = [e_1R] \oplus \dots \oplus [e_kR]$, $k \leq n$. Then from lemma 2.3, each $[e_tR]$ is an ideal. If $[e_tR]$ contains more than one summand then $[e_tR]$ is a simple artinian ring. On the other hand if $[e_tR]$ consists of one summand then e_tR is a local right πc-ring. In this case we prove that e_tR is a right valuation ring. Let C and D be two right ideals of e_tR, such that $C \not\subset D$. Consider $e_tR/(C \cap D)$. Since e_tR is a local right πc-ring $e_tR/(C \cap D)$ is indecomposable π-injective module, and hence uniform. But then $C/(C \cap D) \cap D/(C \cap D) = (0)$, giving $D \subset C$. This proves e_tR is a right valuation ring. The converse is obvious.

<u>Lemma 2.5</u>. Let $\{e_i | i \in I\}$ be an infinite set of orthogonal idempotents of R. Assume for each $J \subset I$, there exists $m(J) \in R$ such that $m(J)e_j = e_j$ for all $j \in J$, and $e_im(J) = 0$ for all $i \in I \backslash J$. Then for all $M_R \supset R_R$, $M/(\sum_{i \in I} e_iR + \ker\theta)$ is not π-injective, where $\theta : R \to \prod_{i \in I} e_iR$, $\theta(r) = \langle e_ir \rangle$ for all $r \in R$.

<u>Proof</u>. See Osofsky [3]. The proof given by her needs no change in order to show that $M/(\sum_{i \in I} e_iR + \ker\theta)$ is not π-injective.

The theorems which follow are analogous to Osofsky's theorem that every cyclic R-module is injective iff R is semisimple artinian. We may remark that if each cyclic module is π-injective then R need not be semisimple artinian. However, we can show that such a ring with $Z(R) = (0)$ is a finite direct sum of rings, each of which is either simple artinian or right Öre-domain. First we prove a key lemma.

Lemma 2.6. Let R be a πc-ring with Z(R) = (0). Then
cannot possess an infinite set of orthogonal idempotents and
is semisimple artinian.

Proof. We follow the technique of Osofsky [3].
ince Z(R) = (0), $\hat{R}$ is regular. Let $\{e_i | i \in I\}$ be a set of
rthogonal idempotents in R. Let $A = \sum_{j \in J} e_j R$ for some $J \subset I$.
ince Z(R) = (0), $cl(A) = \{x \in R | xE \subset A$, for some $E \overset{!}{\subset} R\}$ is
losed in R and is a maximal essential extension of A in R.
ince R is π-injective cl(A) is a direct summand of R. Let
l(A) = m(J)R, m(J) an idempotent in R. Clearly, $m(J)e_j = e_j$
or all $j \in J$. Let $i \in I \backslash J$. Write $\hat{R}e_i m(J) = \hat{R}e$, where e is an
lempotent in $\hat{R}$. Then $m(J)e\hat{R} \cap A = (0)$. Since $m(J)R \overset{!}{\subset} m(J)\hat{R}$,
$\overset{!}{\subset}$ $cl(A) \overset{!}{\subset} m(J)\hat{R}$. Thus $m(J)e\hat{R} = (0)$. Hence $m(J)e = 0$.
is yields $e_i m(J) = e_i m(J)e = 0$. Since R is a right πc-ring
follows from lemma 2.5, that I cannot be infinite. Further,
om proposition 1.5, R and $\hat{R}$ have the same set of idempotents.
nce $\hat{R}$ is semisimple artinian.

Corollary 2.7. A regular right πc-ring is semisimple
tinian.

Corollary 2.8. A self-injective right πc-ring is
miperfect.

In view of corollary 2.8 and theorem 2.4, we have

Theorem 2.9. A ring R is a self-injective right
-ring iff R is a direct sum of A ⊕ B where A is semisimple
tinian and B is a finite direct sum of self-injective right
luation rings.

Theorem 2.10. If R is a right πc-ring with Z(R) = (0) then R = A $\oplus$ B where A is semisimple artinian and B is a finite direct sum of right Öre-domains.

Proof. By lemma 2.6, $\hat{R}$ is semisimple artinian. Write $\hat{R} = R_1 \oplus \ldots \oplus R_n$ where R_i are simple artinian rings. Then by π-injectivity of R, $R = (R_1 \cap R) \oplus \ldots \oplus (R_n \cap R)$ such that $\widehat{(R_i \cap R)} = R_i$ and hence each $R_i \cap R$ is a prime right Goldie ring. Then by theorem 1.16, $R_i \cap R$ is either simple artinian or right Öre-domain.

Second Proof. We can also prove this theorem by invoking theorem 1.15. We can write R = A $\oplus$ B, where A is a regular ring and $\hat{B}$ is strongly regular. Since from lemma 2.6, R (or $\hat{R}$) does not possess any infinite set of orthogonal idempotents, A is semisimple artinian and $\hat{B}$ is finite direct sum of division rings. Since B is π-injective, B is a finite direct sum of right Öre-domains. This completes the proof.

We conclude with an open question. Is it true that a prime right πc-ring has a right zero singular ideal? Since a right valuation ring is clearly a right πc-ring an example of prime right valuation ring which is not a domain shall settle the question in the negative. We are unable to find such an example.

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