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QUASI-INJECTIVE AND PSEUDO-INJECTIVE MODULES

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1. Let R be a ring with identity not equal to zero. A right R-module is said to be quasi-injective (pseudo-injective) if for every submodule N of M, every R-homomorphism (R-monomorphism) of N into M can be extended to an R-endomorphism of M [7] ([13]). An example of a pseudo-injective module which is not quasi-injective was given by Hallett ([4]; also see lemma 2 in this paper). It is known (Harada [5]) that a direct sum of finitely many copies of a quasi-injective module is quasi-injective. We show that if a direct sum of two copies of a pseudo-injective module M is pseudo-injective then M is quasi-injective. Hallett [4] and Singh [14] have shown independently that pseudo-injective modules over PID are quasi-injective. In his thesis, Hallett also showed that self-pseudo-injective generalized uniserial rings are self-injective. In this paper we show that any pseudo-injective module over a generalized uniserial ring is quasi-injective (theorem 4) and we use this to show that any torsion pseudo-injective module over a bounded hereditary noetherian prime ring is quasi-injective (theorem 5). An example is given that a pseudo-injective module over an arbitrary hereditary noetherian prime ring need not be quasi-injective. It is also shown that torsion free pseudo-injective modules over prime Goldie rings are injective and this extends an earlier result of Singh [14].

2. LEMMA 1. *A direct summand of a pseudo-injective module is pseudo-injective.*

The proof is obvious.

THEOREM 1. *Let $N_1 \oplus N_2$ be a pseudo-injective module and $\sigma: N_1 \to N_2$ be a monomorphism. Then σ splits and N_1 is quasi-injective.*

Proof. Since $\eta: \sigma(N_1) \to N_1 \oplus N_2$ given by $\eta(\sigma(x)) = (x, 0)$, $x \in N_1$, is a monomorphism, it can be extended to an endomorphism η^* of $N_1 \oplus N_2$. If $q: N_2 \to N_1 \oplus N_2$ and $p: N_1 \oplus N_2 \to N_1$ are natural injection and projection respectively, then $\lambda = p\eta^* q: N_2 \to N_1$ is such that $\lambda\sigma = I_{N_1}$. Hence σ splits. Let $N_1 \oplus N_1' = N_2$. So $N_1 \oplus N_2 = N_1 \oplus N_1 \oplus N_1'$ and $T = N_1 \oplus N_1$ is pseudo-injective by lemma 1. Write $T = M_1 \oplus M_2$, $M_1 = M_2 = N_1$. Let N be any submodule of N_1 and σ: $N \to N_1$ be an R-homomorphism. If we treat N as a submodule of T contained in M_1, then the mapping

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$\eta: N \to T$ given by $\eta(x)=(x, \sigma(x))$, $x \in N$, is a monomorphism. Hence it can be extended to an endomorphism λ of T. If $q_1: M_1 \to T$ and $p_2: T \to M_2$ are natural injection and projection respectively then $\mu = p_2 \lambda q_1$ is an endomorphism of N_1 which extends σ. Hence N_1 is quasi-injective.

COROLLARY. *A module M is quasi-injective if and only if $M \oplus M$ is pseudo-injective.*

The proof follows from theorem 1 and Harada [5].

3. In this section we study pseudo-injective modules over generalized uniserial rings and hereditary noetherian prime rings (hnp-rings). A right and left artinian ring R is said to be generalized uniserial if for every primitive idempotent e of R eR (Re) have unique composition series as right (left) R-modules. These rings have been called serial by Eisenbud and Griffith [1]. A module X of finite composition length is said to be uniserial if it has a unique composition series. A prime ring R which is left noetherian, left hereditary as well as right noetherian, right hereditary is called a hereditary noetherian prime ring (hnp-ring). For results on hnp-rings we refer to Eisenbud-Robson [2], Lenagan [8], and McConnell-Robson [11].

THEOREM 2. (*Nakayama; see also* [1]). *Let R be a generalized uniserial ring. Then every R-module is a direct sum of uniserial modules.*

The above theorem shows that any indecomposable module over a generalized uniserial ring is a uniserial module. Let E and F be two indecomposable modules over a generalized uniserial ring R and let $m(E, F)$ denote the submodule of E which is minimal among the kernels of all homomorphisms of E into F; as E is uniserial, $m(E, F)$ is well defined and unique. $m(E, F)=0$ if and only if there exists a monomorphism of E into F. For any module X, let $E(X)$ denote its injective hull and $l(X)$ denote the composition length.

THEOREM 3. *Let N be a module over a generalized uniserial ring R. Then N is quasi-injective if and only if $N=\oplus\sum_{i\in\Lambda} N_i$, where N_i are uniserial and $\ell(N_i)\leq l(N_j)+\ell(m(E(N_i), E(N_j)), i, j \in \Lambda$.*

Proof. By theorem 2, $N=\oplus\sum N_i$, where N_i are uniserial. Since R is noetherian, by Matlis [10], $E(N)=\oplus\sum E(N_i)$. For convenience let us write E_i for $E(N_i)$. This gives that any R-endomorphism of $E=E(N)$ is determined by R-homomorphisms between various E_i and E_j. Since a module N is quasi-injective if and only if it is invariant under every endomorphism of $E(N)$ (Johnson and Wong [7]), we obtain that $N=\oplus\sum_{i\in\Lambda} N_i$ is quasi-injective if and only if $\sigma(N_i)\subseteq N_j$ for all $\sigma \in \mathrm{Hom}_R(E_i, E_j)$, $i, j \in \Lambda$. Let N be quasi-injective. Let $\sigma: E_i \to E_j$ be an R-homomorphism with $\ker \sigma = m(E_i, E_j)$. If $N_i \subseteq m(E_i, E_j)$ then obviously $\ell(N_i)\leq \ell(N_j)+\ell(m(E_i, E_j))$; otherwise we must have $m(E_i, E_j)\subset N_i$. Then $N_i/m(E_i, E_j)\cong\sigma(N_i)\subseteq N_j$ gives $\ell(N_i)\leq\ell(N_j)+\ell(m(E_i, E_j))$. Conversely, let the inequality hold. Let

$\sigma \in \operatorname{Hom}_R(E_i, E_j)$. Then using the inequality and the fact that $m(E_i, E_j) \subseteq \ker \sigma$, we immediately get $\sigma(N_i) \subseteq N_j$. Hence N is quasi-injective.

An analogous result for quasi-injective modules over hnp-rings which are not right primitive was established by Singh in [16].

We now use the above theorem to obtain one of our main results.

THEOREM 4. *A pseudo-injective module over a generalized uniserial ring R is quasi-injective.*

Proof. Let N be a pseudo-injective R-module. By theorem 2, we can write $N = \oplus\sum_{i\in\Lambda} N_i$, where N_i are non-zero uniserial modules. Let $E_i = E(N_i)$. We prove that for all $i, j \in \Lambda$, $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$. Then theorem 3 will yield that N is quasi-injective. Clearly we only need to consider the case when $m(E_i, E_j) \subset N_i$. Now by lemma 1, $N_i \oplus N_j$ is pseudo-injective. Let $\sigma: E_i \to E_j$ be an R-homomorphism with $\ker \sigma = m(E_i, E_j)$. Let F_j be the simple submodule of E_j. Since E_j is uniserial, $F_j \subset N_j$. Also then $\sigma^{-1}(F_j) \subset N_i$. Define $\eta: \sigma^{-1}(F_j) \to N_i \oplus N_j$ by $\eta(x) = (x, \sigma(x))$, $x \in \sigma^{-1}(F_j)$. η is an R-monomorphism; thus it can be extended to an R-endomorphism η^* of $N_i \oplus N_j$. If $\lambda_i: N_i \to N_i \oplus N_j$ and $p_j: N_i \oplus N_j \to N_j$ are natural injections and projections, then $p_j \eta^* \lambda_i: N_i \to N_j$ is such that its restriction to $\sigma^{-1}(F_j)$ is equal to the restriction of σ to $\sigma^{-1}(F_j)$. Thus $\ker(p_j \eta^* \lambda_i) = \ker \sigma = m(E_i, E_j)$. Hence

$$N_i/m(E_i, E_j) \cong (p_j \eta * \lambda_i)(N_i) \subseteq N_j$$

gives that $\ell(N_i) \leq \ell(N_j) + \ell(m(E_i, E_j))$. This proves the theorem.

A ring R is said to be right (left) bounded if each of its essential right (left) ideals contains a non-zero two-sided ideal. A ring R which is both right and left bounded is called bounded.

THEOREM 5. *Any torsion pseudo-injective module M over a bounded hnp-ring R, is quasi-injective.*

Proof. Let N be a submodule of M and $\sigma: N \to M$ be an R-homomorphism. We shall show that σ can be extended to an R-endomorphism of M. By an application of Zorn's lemma we suppose that $N \neq M$ and σ cannot be extended to any submodule N' of M containing N properly. Choose $x \in M$ such that $x \notin N$. Now $\operatorname{ann}(x) = \{a \in R \mid xa = 0\}$ is an essential right ideal and so it contains a non-zero two-sided ideal (Eisenbud and Robson [2]). Set $A = \operatorname{ann}(xR)$ which is a non-zero two-sided ideal, and $L = \{y \in M \mid yA = 0\}$. Then L is a module over a generalized uniserial ring R/A. As L is fully invariant submodule of M, L is also pseudo-injective. Hence by theorem 4, L is quasi-injective. Define an R-homomorphism $\lambda: xR \cap N \to L$ by $\lambda(z) = \sigma(z)$, $z \in xR \cap N$. As $xR \subset L$ and L is quasi-injective, λ can be extended to an R-endomorphism λ^* of L. Define $\sigma^*: N + xR \to M$ by $\sigma^*(n + xv) = \sigma(n) + \lambda^*(xv)$. Then σ^* is a well defined R-homomorphism and is a proper extension of σ. This is a contradiction. Hence M is quasi-injective. This completes the proof of the theorem.

REMARK. T. Lenagan [8] has shown that an hnp-ring is either bounded or right as well as left primitive. Thus the above theorem holds for hnp-rings which are not right primitive. The following example due to referee shows that the theorem is not true for an arbitrary hnp-ring.

EXAMPLE. Let Φ be an algebraically closed field and x, y be indeterminates. Let $B=\Phi(y)[x]$ be the hereditary simple principal ideal domain over the field of rational functions $\Phi(y)$ where $xf-fx=df/dy, f\in\Phi(y)$. Then from McConnell and Robson ([11], section 4), we have non-split short exact sequences

$$(1)\qquad 0\to\frac{B}{(x+y)B}\to\frac{B}{x(x+y)B}\to\frac{B}{xB}\to 0$$

$$(2)\qquad 0\to\frac{B}{\left(x+y-\frac{1}{y}\right)B}\to\frac{B}{(x+y)\left(x+y-\frac{1}{y}\right)B}\to\frac{B}{(x+y)B}\to 0$$

such that

$$(3)\qquad \frac{B}{(x+y)B}\approx\frac{B}{\left(x+y-\frac{1}{y}\right)B}(=S,\text{ say})\approx\frac{B}{xB}(=T,\text{ say}).$$

Let $M=B/x(x+y)(x+y-(1/y))B$. Then M is a uniserial module of length 3 having proper submodules

$$M_1=\frac{xB}{x(x+y)\left(x+y-\frac{1}{y}\right)B}\approx\frac{B}{(x+y)\left(x+y-\frac{1}{y}\right)B}$$

and

$$M_2=\frac{x(x+y)B}{x(x+y)\left(x+y-\frac{1}{y}\right)B}\approx\frac{B}{\left(x+y-\frac{1}{y}\right)B};$$

and composition factors T, S, and S. M is pseudo-injective since any monomorphism of M_1 or M_2 is multiplication by an element of Φ (note by theorem 4.1 in [11] that $\text{End}(S)=\text{End}(T)=\Phi$). But M is not quasi-injective since if $\pi:M_1\to(M_1/M_2)\approx M_2$ is a natural homomorphism and $\pi^*\varepsilon\,\text{End}(M)$ is an extension of π, then $(M/\text{Ker }\pi^*)=(M/M_2)\approx M_1$ which is not possible as is clear from (1), (2) and (3).

The above example also provides us an example of a uniserial pseudo-injective module which is not quasi-injective.

4. Next let R be a prime Goldie ring with the $n\times n$ matrix ring D_n over a division ring D as its Öre-ring of quotients. Then by Faith-Utumi's theorem ([3], p. 91) there exists a subring K of R such that $K_n\subseteq R\subset D_n$. It was shown by Singh [14]

that if M is a torsion free pseudo-injective R-module and $1 \in K$ then M is injective. Here we provide a shorter proof of that theorem and do not assume that $1 \in K$.

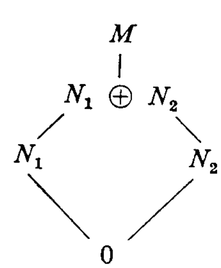
THEOREM 6. *Any torsion free pseudo-injective right module M over a prime right Goldie ring R is injective.*

Proof. Firstly we show that M has a non-zero injective submodule. Let Q be the quotient ring of R. It is well known that MQ is the injective hull of R-module M (Levy [9]), and is a direct sum of isomorphic simple Q-submodules N_i. Then $U=N_i \cap M$ is a uniform R-submodule of M. Let V be a uniform submodule of M which is maximal among uniform submodules of M. We proceed to show that V is quasi-injective. Let W be a submodule of V and $\sigma: W \rightarrow V$ be a non-zero R-homomorphism. Since W, V are uniform and torsion free, σ is an R-monomorphism. Hence it can be extended to an R-endomorphism η of M. Then η is also mono on V. As both V, $\eta(V)$ are essential extensions of $\eta(W)$ in M, $V+\eta(V)$ is also an essential extension of $\eta(W)$ in M. But V is clearly a maximal essential extension of $\eta(W)$ in M and is unique (Johnson [6]). So $V+\eta(V)=V$. Thus $\eta(V) \subseteq V$ and V is quasi-injective. But then V is indeed injective as we proceed to show. Since V is uniform, its injective hull $E(V)$ is isomorphic to eQ where e is a primitive idempotent of Q. We may regard V as a submodule of eQ. Since $eQe \subseteq \mathrm{Hom}_R(eQ, eQ)$ and V is invariant under every R-endomorphism of eQ [7], we get $eQeV \subset V$. Set $U=V \cap R$. Then $QeU=Q$, since $QeU \cap R$ is a non-zero two-sided ideal of R. Hence $V=eQ$ and thus injective. Now by Zorn's lemma we can find a maximal family (M_i) of injective submodules of M such that ΣM_i is a direct sum. Then M must be equal to the direct sum of this family (M_i). Hence M is injective.

REMARK. Suppose R is an hnp-ring which is not right primitive and M is a pseudo-injective R-module which is not torsion. If the torsion submodule $T(M)$ is a direct summand of M then it can be shown that M is injective.

5. An example of a pseudo-injective module over a simple principal ideal domain which is not a quasi-injective module is given in section 3. In this section we give two additional examples of pseudo-injective modules which are not quasi-injective. First we prove

LEMMA 2. *Let M be an R-module whose lattice of submodules is*



and N_1 is not isomorphic to N_2. Then the following hold

(i) *M is not quasi-injective*

(ii) *M is pseudo-injective iff the endomorphism rings of N_i are isomorphic to $Z/(2)$.*

Proof. We first note that any non-zero R-endomorphism f of M is either R-automorphism or has its kernel equal to $N_1 \oplus N_2$. From this it follows that the projection of $N_1 \oplus N_2$ to N_1 (or N_2) cannot be extended to an endomorphism of M. Hence M is not quasi-injective.

To prove (ii) let us assume that endomorphism rings $\mathrm{End}(N_i)$ are isomorphic to $Z/(2)$. Since N_i's are non-isomorphic, any monomorphism of N_1, N_2, or $N_1 \oplus N_2$ to N must be an inclusion map and hence can be lifted to the identity map of M. This shows that M is pseudo-injective. Conversely, let M be pseudo-injective. Let f_1, f_2 be two distinct non-zero members of $\mathrm{End}(N_1)$. Define R-homomorphisms $g_i : N_1 \oplus N_2 \to M$ by $g_i(n_1+n_2)=f_i(n_1)+n_2$, $n_1 \in N_1$, $n_2 \in N_2$, $i=1, 2$. Then g_i is an R-monomorphism and can be lifted to an R-endomorphism h_i of M. Set $h=h_1-h_2$. Then $h \neq 0$, h is not an automorphism and $\ker h \neq N_1 \oplus N_2$. This contradicts the assertion made in the beginning of the proof. Hence $\mathrm{End}(N_1) \cong Z/(2)$. Similarly $\mathrm{End}(N_2) \cong Z/(2)$. This completes the proof of the lemma.

We now give examples of pseudo-injective modules which are not quasi-injective—One due to Hallett and the other due to Teply. It may be verified that each of these modules has its lattice of submodules isomorphic to the lattice in the lemma 2.

Example (Hallett). Let R be an algebra over $Z/(2)$ having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	0	0	n_3	0
e_2	0	e_2	0	n_1	0	0	n_4
e_2	0	0	e_3	0	n_2	0	0
n_1	n_1	0	0	0	0	0	0
n_2	n_2	0	0	0	0	0	0
n_3	0	0	n_3	0	0	0	0
n_4	0	0	n_4	0	0	0	0

Then the right R-module $M=e_2R$ is pseudo-injective but not quasi-injective.

Example (Teply). Let $F=Z/(2)$ and $A=F[X]$. Then $A/(x)$ is a $(A/(x)-A/(x^2))$-bimodule in the natural way, and

$$R = \left\{ \begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \middle| \, u, v \in A/(x),\ w \in A/(x^2) \right\}$$

is a ring with usual binary operations. Let M be the right ideal

$$\left\{ \begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix} \middle| \, v \in A/(x),\ w \in A(x^2) \right\}.$$

Then M_R is pseudo-injective but not quasi-injective.

REMARK. In the above if the field with two elements is replaced by an arbitrary field K then one can verify that the corresponding modules are pseudo-injective if and only if $K \cong Z/(2)$.

We conclude this paper with some open questions.

Question 1. Let M be a right R-module with right singular submodule $M^{\Delta}=(0)$. Is it true that M is pseudo-injective if and only if M is quasi-injective?

The answer is known to be in the affirmative in the following cases: (i) If $2x=0$, $x \in M$ implies $x=0$ (theorem 3.8, [13]), (ii) If R is a prime right Goldie ring (theorem 6, this paper).

Question 2. Let R be a commutative ring (with identity) and M be a pseudo-injective R-module. Is it true that M is quasi-injective?

We can show that if R is commutative artinian then R_R is pseudo-injective iff R_R is injective. We may assume R is local. Then the minimal ideals are all isomorphic and hence by pseudo-injectivity there is only one minimal ideal. Then by Nakayma's definition of QF-rings [12], R is a QF-ring.

(Added Later) Professor Mark L. Teply in [18] has given a construction for forming pseudo-injective modules which are not quasi-injective. He uses this construction to obtain examples which answer in the negative the above questions 1 and 2.

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