

Reprinted from
The Journal of Mathematical Sciences
Volume 2, No. 1, Jan. 1967

ON PSEUDO INJECTIVE MODULES AND SELF PSEUDO INJECTIVE RINGS

SURJEET SINGH and S. K. JAIN

[Received December 31, 1966]

1. Introduction. In this paper we initiate a study of pseudo injective modules and self pseudo injective rings. An R -module M is said to be pseudo injective if and only if every R -isomorphism of each R -submodule of M into M can be extended to an R -endomorphism of M . Johnson and Wong [4] have generalized the concept of injective modules to quasi injective modules, and this is a further generalization of the same. Some of the important results known for quasi injective modules have been proved to hold for pseudo injective modules. It has been shown in theorem (3.4) and (3.8) that under certain conditions a pseudo injective module is quasi injective. Another important result occurs in theorem (3.10) where it is proved that a certain difference module of a pseudo injective module is quasi injective. It still remains open whether every pseudo injective module is quasi injective.

2. Preliminary definitions and notations. Throughout this paper an R -module M will mean a right R -module. An R -module M is said to be injective if for each R -module A and each submodule B of A , every R -homomorphism of B into M can be extended to an R -homomorphism of A into M . An R -module M is said to be quasi injective if each R -homomorphism of each submodule N of M into M can be extended to an R -endomorphism of M . An R -module M is said to be pseudo injective module if and only if each R -isomorphism of each submodule N of M into M can be extended to an R -endomorphism of M .

A ring R is said to be self injective, self quasi injective or self pseudo injective if R is injective, quasi injective or pseudo injective right R -module respectively.

For an R -module M , M^Δ and \hat{M} denote the singular submodule and the injective hull of M respectively. For any ring with $R^\Delta = 0$, \hat{R} denotes the maximal right quotient ring of R as defined by R. E. Johnson in [3]. If $M^\Delta = 0$, then $L'(M)$ denotes the complemented modular lattice of the set of all closed submodules of M . Let A, B be two R -submodules of M ,

then $A \subset' B$ will mean that B is an essential extension of A .

3. Throughout this section it is assumed that M is a pseudo injective R -module with $M^\Delta = 0$ unless otherwise stated.

3.1. LEMMA. Let $N \in L^s(M)$. Then for any R -isomorphism σ of N into M , $\sigma(N) \in L^s(M)$.

Proof. Let $(\sigma(N))^s = K$. Then K is a maximal essential extension of $\sigma(N)$ in M . Define $\sigma' : \sigma(N) \rightarrow M$ such that $\sigma'(\sigma(x)) = x$, for every $x \in N$. Then σ' is an R -isomorphism of $\sigma(N)$ into M . Let $\eta' : M \rightarrow M$ be extension of σ' to M . Let η denote the restriction of η' to K . From the fact that $\ker \eta \cap \sigma(N) = 0$ and K is an essential extension of $\sigma(N)$ it follows that $\ker \eta = 0$. Thus $\eta : K \rightarrow M$ is an R -isomorphism. Then $\eta(K) \cong K$ and $\eta(K) \supseteq N$. Clearly $\eta(K)$ is an essential extension of N . Thus we have $\overline{\eta(K)} = N$, since $N \in L^s(M)$. But also $\eta(\sigma(N)) = N$. Thus we have $K = \sigma(N)$.

3.2. LEMMA. Let $N \in L^s(\hat{M})$. Then for any R -isomorphism σ of N into \hat{M} , $\sigma(N \cap M) = \sigma(N) \cap M$.

Proof. Let $K = N \cap M$. Let $K_1 = \sigma^{-1}\{\sigma(N) \cap M\}$. One can verify that N is an essential extension of K_1 and $\sigma(K_1) \subseteq M$. Then the restriction σ' of σ to K_1 is R -isomorphism of K_1 into M . It can be extended to an R -endomorphism η' of M . Let η be further extension of η' to M . Clearly then the restriction η'' of η to N is such that $(\eta'' - \sigma)K_1 = 0$ and thus $\eta'' = \sigma$, since $K_1 \subset' N$. Hence $\sigma(K) = \eta''(K) = \eta'(K) \subseteq M$. Then

$$\eta''K = \sigma K = \sigma(N \cap M) \subset \sigma(N) \cap M \subset \sigma(N);$$

consequently $\sigma(N) \cap M$ is an essential extension of $\sigma(K)$. But as $\sigma(K) \in L^s(M)$ by (3.1), we obtain $\sigma(N \cap M) = \sigma(N) \cap M$.

3.3. LEMMA. Let σ be an R -isomorphism of \hat{M} into \hat{M} then $\sigma(M) \subseteq M$.

Proof. Since $\hat{M} \in L^s(M)$ by (3.2),

$$\sigma(M) = \sigma(M \cap \hat{M}) = \sigma(\hat{M}) \cap M \subseteq M,$$

completing the proof.

3.4. THEOREM. Let M be any R -module with $M^\Delta = 0$ and $L^s(M)$ be finite dimensional. Then M is quasi injective if and only if M is pseudo injective*.

Proof. Sufficiency. Let M be pseudo injective R -module with $M^\Delta = 0$. Now as each member of $\text{Hom}_R(M, M)$ can be uniquely extended to member of $\text{Hom}_R(\hat{M}, \hat{M})$, we shall regard $\text{Hom}_R(M, M) \subseteq \text{Hom}_R(\hat{M}, \hat{M})$. Since $L^s(M)$ is finite dimensional therefore as a special case of [1, theorem 1.12] we get $\text{Hom}_R(M, M)$ is a semi-simple ring with descending chain condition. It is not difficult to prove that each element of a semi simple ring with d.c.c. can be expressed as a sum of finite member of unit elements. Now because of (3.3) each unit element of $\text{Hom}_R(\hat{M}, \hat{M})$ is in $\text{Hom}_R(M, M)$. Hence we have $\text{Hom}_R(\hat{M}, \hat{M}) \subseteq \text{Hom}_R(M, M)$. Consequently

$$\text{Hom}_R(\hat{M}, \hat{M}) = \text{Hom}_R(M, M).$$

Thus it follows from [4, theorem 1.3] that M is quasi injective. Necessity is obvious.

3.5. LEMMA. Let M be an injective R -module with $M^\Delta = 0$ $L^s(M)$ be finite dimensional. If

$$\begin{aligned} M &= N_1 \oplus N_2 \oplus \dots \oplus N_t, \text{ and also} \\ &= K_1 \oplus K_2 \oplus \dots \oplus K_m \end{aligned}$$

be two decompositions of M as a direct sum of atoms of $L^s(M)$ then $t = m$ and there exists one-one correspondence between the N_i and K_j so that the corresponding atoms are isomorphic as R -modules.

Proof. Let M be an injective R -module with $M^\Delta = 0$ and $L^s(M)$ finite dimensional. We know that $L^s(M)$ is complemented modular lattice with 0 and 1. Each $A \in L^s(M)$ is injective. For $A, B \in L^s(M)$, $A \vee B = (A+B)^s$. But as A, B closed imply that $A + B$ is also injective and hence closed. Consequently $A \vee B = A + B$. Then by a theorem of Kurosh and Ore [6, page 204] on modular lattices, if

$$M = N_1 \oplus N_2 \oplus \dots \oplus N_t \text{ and } M = K_1 \oplus K_2 \oplus \dots \oplus K_m$$

are two decompositions of M as direct sum of atoms of $L^s(M)$ then $t = m$, and there is one to one correspondence between the N_i and K_j such that the corresponding atoms are directly projective [6, page 204]. Without loss of generality we can suppose that K_i and N_i for each i are directly projectly projective. Hence there exist $L_i \in L^s(M)$ such that

$$L_i + K_i = L_i + N_i$$

*In fact the theorem is true even if $L^s(M)$ is only atomic.

and $L_i \cap N_i = L_i \cap K_i = 0$. But then

$$N_i \cong N_i/N_i \cap L_i \cong N_i + L_i/L_i = K_i + L_i/L_i \cong K_i/K_i \cap L_i \cong K_i.$$

Hence $N_i \cong K_i$.

From (3.5) the following is immediate.

3.6. LEMMA. Let M be any injective module with $M^\Delta = 0$ and $L^s(M)$ finite dimensional. If N, N_1 be two injective submodules of M which are isomorphic to each other and further if

$$\hat{M} = N \oplus K = N_1 \oplus K_1$$

then K is isomorphic to K_1 .

3.7. THEOREM. If $M^\Delta = 0$ and $L^s(M)$ finite dimensional then M is pseudo injective if and only if M is invariant under every R -isomorphism of \hat{M} into \hat{M} .

Proof. Necessity follows from (3.3). For sufficiency let M be invariant under every R -isomorphism of \hat{M} into \hat{M} . Let N be a submodule of M and σ any isomorphism of N into M . Then σ can be extended to an R -isomorphism σ' of the injective hull \hat{N} of N into \hat{M} . Now we can express $\hat{M} = \sigma'(\hat{N}) \oplus K_1 = \hat{N} \oplus K$. Then by (3.6) there exists an R -isomorphism η of K onto K_1 . Then the mapping $\lambda : \hat{M} \rightarrow \hat{M}$ defined as $\lambda(x+y) = \sigma'(x) + \eta(y)$; $x \in \hat{N}, y \in K$ is an R -isomorphism of \hat{M} onto \hat{M} . Consequently, by supposition $\lambda(M) \subseteq M$. Hence the restriction of λ to M is desired extension of σ . Hence \bar{M} is pseudo injective.

Because of theorem (3.4) we can replace 'pseudo injective' by 'quasi injective' in theorem (3.7).

3.8. THEOREM. If M be a pseudo injective module with $M^\Delta = 0$ such that there exist two consecutive positive integers, n and $(n+1)$ such that for any $x \in M, nx = 0$ implies $x = 0$ and also $(n+1)x = 0$ implies $x = 0$, then M is quasi injective.

*Proof.** Let M be a pseudo injective module with the given hypothesis. Let N be any submodule of M , and σ any R -isomorphism of N into M . Now there exist submodule K of N such that $K \cap \ker \sigma = 0$ and $K + \ker \sigma$ is essential in N .

Let σ' be restriction of σ to K then σ' is an R -isomorphism of K into M . Let η be an extension of σ' to M . Define a mapping $\xi' : (K + \ker \sigma) \rightarrow M$

*The proof of the theorem for $n = 1$ was communicated to the authors by R. E. Johnson.

by $\xi'(x+y) = nx - y$; $x \in K, y \in \ker \sigma$. Because of the hypothesis one can see that ξ is an R -isomorphism of $K + \ker \sigma$ into M . So let ξ be an extension of ξ' to an R -endomorphism of M . Let $N' = \{(n+1)x \mid x \in M\}$, N' is a submodule of M . Consider the mapping $\lambda: N' \rightarrow M$ defined as $\lambda((n+1)x) = x$ for every $x \in M$. This mapping is well defined and is an R -isomorphism of N' into M . It can be extended to an R -endomorphism λ' of M . But then if for any $x \in M, \lambda'(x) = y$ then

$$\lambda'((n+1)x) = (n+1)y.$$

But $\lambda'((n+1)x) = \lambda((n+1)x) = x$. Hence $(n+1)y = x$. Thus

$$\lambda'(x) = \frac{1}{n+1}x.$$

Then $\lambda'(\eta\xi + \eta); M \rightarrow M$ is such that $\lambda'(\eta\xi + \eta)(z) = \sigma(z)$ for every $z \in N$. Hence $\lambda'(\eta\xi + \eta)$ is an extension of σ . This shows that M is quasi injective.

By torsion free module M we mean that $kx = 0, k$, a non-zero integer, $x \in M$ if and only if $x = 0$.

3.9. COROLLARY. Any torsion free module M which is pseudo injective is quasi injective.

Next, let $Q = \{x \in M \mid 2^n x = 0 \text{ for some +ve integer } n\}$. Let $P = Q^s$, the closure of Q . Let $\bar{M} = M - P$.

3.10. THEOREM. If M is pseudo injective with $M^\Delta = 0$ then \bar{M} is quasi injective.

Proof. Clearly for any R -homomorphism σ of a submodule of M into $M, \sigma(N \cap Q) \subseteq Q$. Now we prove that $\sigma(N \cap P) \subseteq P$. Let $x \in N \cap P$ then $x \in P$ and therefore $xE \subseteq Q$ for some essential right ideal E . Then $\sigma(xE) \subseteq Q$. Thus $\sigma(x)E \subseteq Q$. This implies $\sigma(x) \in P$. Let M be a pseudo injective module. We firstly prove that \bar{M} is also pseudo injective. Let σ be any R -isomorphism of a submodule $\bar{N} = N - P$ of \bar{M} into \bar{M} . Let $\sigma(\bar{N}) = K - P$. Let K_1 be complement of P in K . Then $K_1 + P \subset' K$ and $(K_1 + P) - P \subset' K - P$. Therefore $\sigma^{-1}[(K_1 + P) - P] \subset' \bar{N}$. Let $\sigma^{-1}[(K_1 + P) - P] = \bar{N}_1 = N_1 - P$. Let N_2 be complement of P in N_1 . Then $N_2 + P \subset' N_1$ and $(N_2 + P) - P \subset' \bar{N}$. This all gives

$$(N_2 + P) - P \subset' \bar{N} \text{ and } [(N_2 + P) - P] \subseteq [(K_1 + P) - P]$$

Define a mapping $\eta; N_2 + P \rightarrow M$ by $\eta(x+y) = x' + y$; $x \in N_2, y \in P, x' \in K$ is such that $\sigma(x) = x'$. To show that the mapping is well defined. Let $x + y = 0$, then $x = 0$, since $N_2 \cap P = 0$. Hence $x = 0$.

Consequently $\sigma(\bar{x}) = \bar{x}' = 0$. This gives $x' \in K_1 \cap P = (0)$. Therefore $x' = 0$. Hence $x' + y = 0$. This shows that η is well defined. To show that η is one-one let for some $x \in N_2, y \in P, \sigma(x+y) = x' + y = 0$. That gives $x' = 0 = y$. Hence $\bar{0} = \bar{x}' = \sigma(\bar{x})$. Since σ is one-one, so $\bar{x} = \bar{0}$. Consequently $x \in N_2 \cap P = (0)$. That gives $x = 0$. Hence $x + y = 0$. Hence η is also one-one. Now M is given to be pseudo injective, therefore η can be extended to an R -endomorphism $\bar{\xi}$ of \bar{M} . Define $\bar{\xi}: \bar{M} \rightarrow \bar{M}$ as follows $\bar{\xi}(\bar{x}) = \bar{\xi}(x)$. The mapping is well defined, since $\bar{x} = \bar{0}$ implies $x \in P$ and hence as is proved earlier $\bar{\xi}(x) \in P$, therefore $\bar{\xi}(\bar{x}) = \bar{0}$. Clearly $\bar{\xi}$ is an R -endomorphism of \bar{M} . The way we have constructed $\bar{\xi}$, it shows that $\bar{\xi}(\bar{x}) = \sigma(\bar{x})$ for every $\bar{x} \in (N_2+P) - P$. But as

$$(N_2+P) - P \subset' \bar{N}, \text{ and } \bar{M}^\Delta = 0.$$

Consequently $\bar{\xi}(\bar{x}) = \sigma(\bar{x})$ for every $\bar{x} \in \bar{N}$. Hence $\bar{\xi}$ is an extension of σ . Hence \bar{M} is pseudo injective. In view of the theorem (3.8) it is enough to show that for any $\bar{x} \in \bar{M}, 2\bar{x} = \bar{0}$ implies $\bar{x} = \bar{0}$. Since for $n = 1$, then \bar{M} satisfies the condition of theorem (3.8). Let $2\bar{x} = \bar{0}$, then $2x \in P$ and therefore $(2x)E \subseteq Q$ for some essential right ideal E of R . Thus for each $c \in E, 2^n(2xc) = \bar{0}$ for some n . This implies $2^{n+1}xc = 0$, and $xc \in Q$. Consequently $xE \subseteq Q$, and $x \in P$. This implies $\bar{x} = \bar{0}$. Hence the theorem follows.

4. In this section we will prove some results for pseudo injective modules which are known to hold for quasi injective modules and their proofs are almost on the same line as those for quasi injective modules. Modules considered here are pseudo injective modules. For any ring $R, J(R)$ denotes the Jacobson radical of R .

4.1. THEOREM. If M is a uniform R -module with $M^\Delta = 0$ then $\text{Hom}_R(M, M)$ is a division ring.

Proof. The proof is immediate from (3.1) and thus omitted.

4.2. THEOREM. For any pseudo injective module M , if $E = \text{Hom}_R(M, M)$ then $J(E) = \{\alpha \in E \mid \ker \alpha \subset' M\}$, and $E - J(E)$ is Von-Neumann regular.

Proof. Let $I = \{\alpha \in E \mid \ker \alpha \subset' M \text{ if } \alpha, \beta \in I, \text{ then}$

$$\ker(\alpha - \beta) \supseteq \ker \alpha \cap \ker \beta$$

But $\ker \alpha \subset' M$ and $\ker \beta \subset' M$ imply $\ker \alpha \cap \ker \beta \subset' M$.

Therefore $\alpha - \beta \in I$

Further for any $\lambda \in E$, $\ker \lambda \subset M$, since, $\ker \alpha$ contained in $\ker \lambda$ is essential in M . Therefore $\lambda \alpha \in I$. Hence I is a left ideal of E . Now since $\ker \alpha \cap \ker (1+\alpha) = 0$ for any $\alpha \in I$ and $\ker \alpha \subset M$, therefore $\ker (1+\alpha) = 0$. If $N' = (1+\alpha)M$, then the mapping $\beta: N' \rightarrow M$, where $\beta((1+\alpha)x) = x$ is an isomorphism of N' into M . Thus it can be extended to an R -endomorphism γ of M . Then γ is a left inverse of $(1+\alpha)$. Consequently each member of I is left quasi regular. Hence $I \subseteq J(E)$. Now we show that for each $\lambda \in E$, there exist $\theta \in E$ such that $\lambda\theta\lambda - \lambda \in I$. Let L be a relative complement of $K = \ker \lambda$. Then the mapping $\lambda x \rightarrow x$, for $x \in L$ is an R -isomorphism of the submodule λL into M . Since M is pseudo injective, so it can be extended to an R -endomorphism θ of M . Then $\ker(\lambda\theta\lambda - \lambda) \supseteq K + L$, and we know that $K + L \subset M$. Consequently $\lambda\theta\lambda - \lambda \in I$.

Now we show that $J(E) = I$. For any $\lambda \in J(E)$ we can choose a $\theta \in E$ such that $u = \lambda - \lambda\theta\lambda \in I$. But $-\lambda\theta \in J(E)$ since $J(E)$ is an ideal and therefore $(1-\lambda\theta)^{-1}$ exists. Therefore $(1-\lambda\theta)^{-1}u = \lambda$ and $\lambda \in I$, since I is a left ideal. Thus $J(E) = I$ as asserted. From what we have done above it also follows that $E - J(E)$ is a Von-Neumann regular ring.

4.3. COROLLARY. If $M^\Delta = 0$ then $\text{Hom}_R(M, M)$ is Von-Neumann regular.

Proof. Let $E = \text{Hom}_R(M, M)$. By theorem (4.2) if $\alpha \in J(E)$ then $\ker \alpha \subset M$, but then $\alpha = 0$, since $M^\Delta = 0$. Consequently $J(E) = 0$. Hence E is Von-Neumann regular.

5. In this section we consider a ring R which is self pseudo injective.

5.1. THEOREM. If R is a self pseudo injective with $R^\Delta = 0$, then $E = \text{Hom}_R(R, R)$ is a Von-Neumann regular ring which is also self pseudo injective.

Proof. E is Von-Neumann regular follows from (4.3). We now prove that E is also self pseudo injective. Since R is isomorphic to the ring of left multiplications on R given by elements of R , we shall regard $R \subseteq E \subseteq \hat{R}$.

Let I be any right ideal of E and σ any E -isomorphism of I into E . Let $I' = \sigma^{-1}[\sigma(I) \cap R]$. Then I' is an essential R -submodule of I and $\sigma(I') \subseteq R$. Let σ' be restriction of σ to I' . Then σ' can be extended to an R -endomorphism η' of R . Let η' be extended to \hat{R} -endomorphism η of \hat{R} . Since $\text{Hom}_R(\hat{R}, \hat{R}) = \text{Hom}_{\hat{R}}(\hat{R}, \hat{R})$, therefore η is also an \hat{R} -endomorphism of \hat{R} . Thus η is also an E -endomorphism of \hat{R} . For any

$$\lambda \in E, r \in R, (\eta\lambda)r = \eta(\lambda r) = \eta'(\lambda r) \in R,$$

SURJEET SINGH AND S. K. JAIN

Hence $\eta\lambda \in E$ for every $\lambda \in E$, showing that the restriction of η to E is an E -endomorphism of E . Now I as an R -module is such that the restriction ξ' of ξ to I and σ coincide on I as R -homomorphisms of I into E . Thus $\xi' = \sigma$, since $I' \subset I$ as R -modules. Hence ξ is the desired extension of σ . Hence E is self pseudo injective.

LEMMA. Any ring with $R^\Delta = 0$.

Let $\{x \in R : 2^n x = 0 \text{ for some positive integer } n\}$ and $P =$ closure of this set. P is a two sided ideal and R/P is a ring with zero singular

LEMMA. If R be any self pseudo injective ring with $R^\Delta = 0$ and d any element, then R/P is self injective.

From theorem (3.10), it follows that R/P is a quasi-injective ring. Let $\bar{I} = I/P$ be any right ideal of R/P and $\sigma : \bar{I} \rightarrow R/P$ an R -homomorphism of \bar{I} into R/P . \bar{I} is also a right R -submodule of R/P . Hence it can be extended to an R -homomorphism of \bar{I} into R/P . Hence it can be extended to an R -endomorphism η of R/P . But then clearly η is also a R/P -endomorphism of \bar{I} . Hence R/P is self quasi-injective module. But we know that R/P is a self pseudo injective ring. Consequently R/P is also self injective.

LEMMA. Let R be a ring with $R^\Delta = 0$ and d any element of R having $dR = 0$. Then the left multiplication by d gives an R -homomorphism of R into R . Then from (3.3) the following is immediate.

LEMMA. Let R be a ring with $R^\Delta = 0$ and d any element of R having $dR = 0$. Then the left multiplication by d gives an R -homomorphism of R into R . Then from (3.3) the following is immediate.

LEMMA. If R is a self pseudo injective ring with $R^\Delta = 0$, then $dR = 0$, $dR \subseteq R$.

LEMMA. If R is a self pseudo injective ring with $R^\Delta = 0$ and \hat{R} its quotient ring then $\hat{R} = R$.

PROOF. This is easy and is therefore omitted.

LEMMA. If R is a ring with $R^\Delta = 0$ such that \hat{R} is a classical ring of quotients of R then R is self injective if and only if R is self pseudo injective.

PROOF. This follows from (5.4).

REFERENCES

- FAITH C. and UTUMI Y., Quasi-injective modules and their endomorphism rings, *Arch. Math.* 15 (1964) 166-174.
- FAITH C. and CHASE S. U., Quotient rings and direct products of full linear rings, *Math. Z.* 88 (1965) 250-264.
- JOHNSON R. E., The extended centralizer of a ring over a module, *Proc. A.M.S.* 2 (1951) 891-895.
- JOHNSON R. E. and WONG E.T., Quasi injective modules and irreducible rings, *J. Lon. Math. Soc.* 36 (1961) 260-268.
- JOHNSON R.E., Structure theory of faithful rings II, *Trans. A.M.S.*, 54 (1957) 523-544.
- JACOBSON N., Lectures in abstract algebra, Vol. I, 1951.

rori Mal College, Delhi,
 Faculty of Mathematics, University of Delhi, Delhi-7, India.