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RINGS WHOSE CYCLIC MODULES ARE INJECTIVE OR PROJECTIVE

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ABSTRACT. The object of this paper is to prove

Theorem. *For a ring R the following are equivalent*:

(i) *Every cyclic right R-module is injective or projective.*

(ii) *$R = S \oplus T$ where S is semisimple artinian and T is a simple right semihereditary right Öre-domain whose every proper cyclic right module is injective.*

Faith [2] called a ring R a right PCI-ring if each cyclic right R-module $C \not\approx R$ is injective, and has shown that a right PCI-ring is either semisimple artinian or else a simple right semihereditary right Öre-domain. The proof of our Theorem provides an alternative and a shorter proof of Faith's result that if R is a regular right PCI-ring (equivalently if R is not a right PCI-domain), then R is semisimple artinian. The existence of PCI-domains which are not division rings is given by Cozzens [1].

Throughout the lemmas the ring R satisfies condition (i) of the Theorem.

Lemma 1. *Let I be a two-sided ideal of R. Then R/I satisfies* (i). *Further, if R/I is injective, then R/I is semisimple artinian.*

Proof. Clear by Osofsky [3].

Lemma 2. *R does not contain an infinite set of central orthogonal idempotents.*

Proof. Let $\{e_i\}_{i\in\Lambda}$ be an infinite set of central orthogonal idempotents and Λ' be an infinite subset of Λ such that $\Lambda - \Lambda'$ is infinite. Set $A = \bigoplus\sum_{i\in\Lambda} e_i R$, $B = \bigoplus\sum_{i\in\Lambda-\Lambda'} e_i R$. R/B cannot be projective since B is infinitely generated. Hence R/G is semisimple artinian. But A/B is an infinitely generated ideal in R/B. This yields a contradiction.

Lemma 3. *For any idempotent e in R, eR or $(1-e)R$ is a completely reducible injective right R-module.*

Proof. First we show that the right singular ideal $Z(R) = 0$.

If $a \in Z(R)$ then $aE = 0$ where E is some essential right ideal of R.

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This implies $aR \cong R/E$. If aR is projective then $E = R$ and so $a = 0$. If aR is injective then $aR = eR \subset Z(R)$, $e^2 = e$. This again implies $aR = 0$ and thus $a = 0$. Now write $R = eR \oplus (1 - e)R$. If both eR and $(1 - e)R$ are completely reducible then the result is obvious. So suppose eR is not completely reducible. Then there exists a proper essential R-submodule A of eR. eR/A is then a singular R-module and since $Z(R) = (0)$, it cannot be projective. So from $R/A \cong eR/A \oplus (1 - e)R$, we get R/A must be injective. Hence $(1 - e)R$ is injective. If $(1 - e)R$ were also not completely reducible, then, as before, we get eR is injective, and hence R is self-injective. Then Lemma 1 yields that R is semisimple artinian and we are done. Thus in any case if eR is not completely reducible, then $(1 - e)R$ is a completely reducible injective module.

Lemma 4. *Either R is an integral domain or R has a nonzero socle.*

Proof. By Lemma 3 if R has an idempotent $e \neq 0, 1$, then R has a nonzero socle. Now suppose R dose not possess idempotents different from 0 and 1. Let $0 \neq a \in R$ and $r(a) = \{x \in R | ax = 0\}$. Then $aR \cong R/r(a)$. If aR is injective then $aR = (0)$ or $aR = R$. The former implies $a = 0$ and the latter implies R is right self-injective and hence semisimple artinian, consequently a division ring. If aR is projective then $r(a) = (0)$ or $r(a) = R$. The latter is not possible. Hence $r(a) = 0$ and R is an integral domain.

Lemma 5. *If R has no nontrivial central idempotents then either R is simple artinian or R is a simple right semihereditary right Öre-domain.*

Proof. If R is a domain then R is a right PCI-ring and hence, by Faith [2, Propositions 5, 17], R is a simple right semihereditary right Öre-domain. If R is not a domain then R has a nonzero socle S. From Lemma 3 every minimal right ideal of the form eR, $e = e^2 \in R$, is injective. Thus hypothesis (i) yields that every minimal right ideal aR of R is generated by an idempotent. But then it follows immediately that R is semiprime. Indeed R can be shown to be prime since R has no nontrivial central idempotents. In case R/S is projective, then $S = eR$ where e is a central idempotent, so that $R = S$ is simple artinian. If R/S is injective then R/S is semisimple artinian, and hence R is regular. Thus R is a primitive regular ring with nonzero socle. Let $\hat{R}$ denote the maximal right quotient ring of R. Since every minimal right ideal is injective, the socle of $\hat{R}$ = socle of $R = S$. If S is finitely generated then $S = R = \hat{R}$ and hence R is simple artinian. So assume that S is not finitely generated. Then there exists right ideals K_1, K_2 in S such that $S = K_1 \oplus K_2$ and $S \approx K_1 \approx K_2$. Since K_1 is infinitely generated, R/K_1 is injective. Also R/K_1 contains $(K_1 \oplus K_2)/K_1 \approx K_2 \approx S$. Hence $\hat{R}_R = \hat{S}_R$ is embeddable in R/K_1. This implies $\hat{R} = xR$ for some $x \in \hat{R}$, so there exists $a \in R$ such that $xa = 1$. This implies $a + S$ is invertible in the semisimple artinian ring R/S. Thus there exists $y + S$ in R/S such that $ay - 1 \in S$. This yields that $x \in R$. Hence $R = \hat{R}$ which by Lemma 1 implies R is simple artinian. This proves the lemma.

Proof of the Theorem. Assume (i). By Lemma 2 we can write $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ where R_i are rings having no nontrivial central idempotents. Then by Lemma 3 we get all R_i, excepting at most one, are semisimple artinian. Hence either R is semisimple artinian or $R = S \oplus T$ where S is semisimple artinian and T has no nontrivial central idempotents. Since T also satisfies (i), Lemma 5 completes the proof. The converse is clear.

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