RINGS WITH A POLYNOMIAL IDENTITY

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Section 1

Let R be a ring which is an algebra over some commutative ring F such that $\alpha x = 0$, $\alpha \in F$ and $x \in R$, implies $\alpha = 0$ or $x = 0$. R said to satisfy a polynomial identity if there exists a non-zero element $p(x_1, \ldots, x_n)$ of a free algebra $F[x_1, \ldots, x_n]$ over F in non-commuting indeterminates x_i such that $p(a_1, \ldots, a_n) = 0$ for all $x_i = a_i \in R$. R is defined to be left rationally complete if R as a left R-module does not possess a proper rational extension in the sense: ${}_RA$ is a rational extension of ${}_RR$ in case for each submodule ${}_RB$, $A \supseteq B \supseteq R$, $f \in \mathrm{Hom}\,(B,A)$ satisfies $f(R) = 0$ iff $f = 0$. (Equivalently $\mathrm{Hom}_R(B - R,A) = 0$). It is known R is left rationally complete means that R is its own Utumi's left ring of quotients (see Faith [3]). The left singular ideal $Z({}_RR)$ of R is the two-sided ideal consisting of $x \in R$ such that the left annihilatorof x is essential left ideal. If R is left rationally complete and $Z({}_RR) = 0$ then R is known to be von-Neumann regular. (See Faith [3]). Our first result is:

<u>THEOREM 1</u>. Let R be semi-prime, left rationally complete and satisfies a polynomial identity. Then $R = S \times T$ where S is a product of finite dimensional central simple algebras and T is von-Neumann regular with zero socle and satisfies a polynomial identity.

Proof. It is shown by Fisher [4] that $Z({}_RR) = 0$. Let X be the socle of R. Set $Y = \{a \in R \mid Ea \subset x$, where E is some essential left ideal in $R\}$ then Y is the maximal essential extension in R. It is known that if $Z({}_RR) = 0$ then a left R-module M is a rational extension of R iff it is an essential extension of $_RR$. But then since R is left rationally complete R as left R-module cannot possess any proper essential extension which implies $_RR$ is injective (prop. 5, P. 59, Faith [3]). Hence Y is a direct summand of R as left R-module. Let $Y = Re$. Further R being left self-injective and with left zero singular ideal, R is von-Neumann regular (Faith [3], p. 69). This implies e is central idempotent because Y is a two sided ideal. So we have $R = Re \oplus R(1 - e)$ where e is central. Set $S = Re$. S is a regular ring with PI and its socle X is essential as a left ideal. We know the $X = \oplus \sum X_i$ where X_i, the homogeneous components of X are simple rings, and also X_i's are invariant under any X-homomorphism of X. Further since X_i are simple PI-rings, they are full matrix rings and hence possess identity elements. So we have $\mathrm{Hom}_X(X,X) = \prod \mathrm{Hom}_X(X_i, X_i)$

$= \prod \mathrm{Hom}_{X_i}(X_i, X_i)$

$= \prod X_i$, the product of simple rings with PI

= The product of finite dimensional central simple algebras.

Since X is essential left ideal in S, the left quotient ring of $X = S$. But the maximal quotient ring of $X = \bigcup_E \mathrm{Hom}_X(E,X)/\equiv$ where the union runs over all essential left ideals E of X and the relation $\equiv$ is defined as: two maps are equivalent iff they agree on some essential left ideal. Indeed the only essential left ideal in X is X itself. Hence we get $S = \mathrm{Hom}_X(X,X)$, which is a product of finite dimensional central simple algebras. Next set $T = R(1 - e)$. Then T is clearly regular ring with a polynomial identity. If A is a minimal left ideal in T then A is a minimal left in R. But then $A \subset Re$, a contradiction. Hence T must have zero socle. This proves the theorem.

Corollary (Exercise 7, P. 46, Lambek [8]). Let R be commutative semi-prime and rationally complete. Then $R = S \times T$ where S is a direct product of fields and the Boolean Algebra of the annihilator ideal of T(= the Boolean Algebra of the central idempotents of T) has no atoms. Equivalently T has zero socle.

Proof. Follows directly from the theorem, since the only commutative finite dimensional central simple algebras are fields.

AN EXAMPLE. We give an example of a von-Neumann regular ring with a polynomial identity which is rationally complete but has zero socle. Take A to be the Boolean Algebra which is not rationally complete (for example the set of all finite and cofinite sets of natural numbers form such a Boolean algebra, see Lambek [8], P. 45). Let Q(A) be the rational completion of A. Then Q(A) is also Boolean and hence commutative and von-Neumann regular. By the theorem Q(A) = SxT where T has zero socle and is a von-Neumann regular ring with a polynomial identity. T cannot be zero. For then Q(A) = direct product of fields each with two elements. Each of these fields F, being an ideal of Q(A), must have non-zero intersection with A because Q(A) is rational completion of A. But then $F \subseteq A$, since F has only two elements. Hence A = Q(A). This contradicts that A is not rationally complete.

Section II

A set of pre-equivalence data (all called Morita context) consists of C-algebras R and S, bimodules ${}_RV_S$ and ${}_SW_R$, and bimodule homomorphisms $f : V \otimes_S W \to R$ and $g : W \otimes_R V \to S$ which are associative in the following sense: Writing $f(v \otimes w) = vw$ and $g(w \otimes v) = wv$, we require: (i) $(vw)v' = v(wv')$ and (ii) $(wv)w' = w(vw')$ for all $v, v' \in V$ and $w, w' \in W$. We have been investigating how some of the properties of the ring R influence the ring S. In this note we give one such simple result which though easy seems to have interesting consequences. We assume that f and g are not zero maps.

PROPOSITION. If R satisfies some polynomial identity there exists a non-zero right ideal in S which satisfies a polynomial identity, i.e., S satisfies a generalized polynomial identity.

Proof. Choose $w \in W$ such that $wV \neq 0$. Suppose $\sum \alpha_i x_{i_1} s_{i_2} \cdots x_{i_n} = 0$ is a multilinear identity satisfied by R. Picking $v_i w$, $1 \leq i \leq n$, in R, we have

$$\sum \alpha_i v_{i_1} w v_{i_2} w \cdots v_{i_n} w = 0$$

This gives

$$\sum \alpha_i w v_{i_1} w v_{i_2} \cdots v_{i_n} w v = 0$$

Hence wV, a right ideal in S, satisfies the identity

$$\sum \alpha_i x_{i_1} x_{i_2} \cdots x_{i_n} x = 0.$$

By symmetry we can also find a non-zero left ideal in S satisfying a polynomial identity.

Corollary. If R is a prime ring with a polynomial identity, V is a torsionless left R-module such that $d({}_R V)$ is finite and (R, V, V^*, E, f, g) is the natural Morita context then $E = \mathrm{Hom}_R(V,V)$ satisfies a polynomial identity.

Proof. We note by a result of Zelmanowitz [9] that E is a prime Goldie ring. Hence we are done by using the result of Belluce - Jain that if there exists a non-zero one-sided ideal with PI in a prime Goldie ring, then the whole ring has PI.

By appealing to the result of Amitsur ([1], p. 291) we have also for a general Morita context the following

Corollary. Let R be a semiprime ring with acc on annihilator two sided ideals and satisfying a polynomial identity (Equivalently let R be a semiprime Goldie ring with a polynomial identity). Let V be a torsionless left R-module and (R, V, W, S, f, g) be a Morita context. Then S contains a non-zero left ideal with a polynomial identity and if $d({}_R V) < \infty$ and ${}_S W$ is torsion free then S itself satisfies a polynomial identity.

In case the ring S satisfies a polynomial identity and has same multilinear identities as those of a one-sided ideal then the following can be shown.

LEMMA. Let S be a ring satisfying a polynomial identity and S as left S-module is faithful. Let A be a non-zero right ideal in S. Suppose that the T-ideals (the ideals of identities) of S and A are equal. Then A contains a non-zero two-sided ideal of S.

Proof. Let d be the minimal degree of a polynomial identity satisfied by S. We know that we can then find a multilinear polynomial identity of the degree d. Let this polynomial identity be $p(x_1, \ldots, x_d) = 0$. Rewrite it as

(1) $x_1q_1 + x_2q_2 + \ldots + x_dq_d = 0$

where q_i are polynomials of degree $\leq d - 1$, and no q_i involve x_i. Since A and S have the same T-ideal, (1) is also a polynomial identity of minimal degree for A. Choose $a_2, a_3, \ldots, a_d$ in A (not all of them if not needed) such that $q_1 = q_1(a_2, \ldots, a_d)$ is not zero. Then substituting $x_2 = a_2, \ldots, x_d = a_d$ (picking arbitrary non-zero element of A for that x_i for whom a_i was not obtained before when the choices of $a_2, \ldots, a_d$ were made) and $x_1 = r \varepsilon R$, we get $ra \varepsilon A$ where $a = q_1(a_2, \ldots, a_d) \neq 0$, and r is any element in S. Thus $Sa \subseteq A$ and a generates a non-zero two-sided ideal in A with polynomial identity as desired.

REMARK. The lemma holds with the weaker assumption that A and S satisfies the same multilinear identities.

We proceed to give an interesting application of this lemma. But we first state the known results in [2] and [5] which we shall need.

THEOREM. (Belluce-Jain [2]) Let R be a prime ring. If A is a non-zero right ideal satisfying a polynomial identity and $\ell(A) = 0$, then A and R satisfies the same multilinear identities.

THEOREM (Jain [5]). Let R be a prime ring. Then a non-zero one-sided ideal in R has a polynomial identity iff R is a special Johnson ring (in particular R has both left and right singular ideals zero).

REMARK. We should remark that though it is not so explicitly stated in [2], theorem 1) A and R satisfies same multilinear identities, but this fact is explicit in the proof. (See also theorem 3.1 in [6]).

The above results yield

THEOREM. The following are equivalent for a prime ring R (regarded as an algebra over its centroid):

(1) R satisfies a polynomial identity.

(2) There exists a non-zero right ideal A in R such that $\ell(A) = 0$ and A satisfies a polynomial identity.

(3) There exists a non-zero left ideal A in R such that $r(A) = 0$ and A satisfies a polynomial identity.

(4) There exists a non-zero two-sided ideal in R with a polynomial identity.

(5) There exists an essential right ideal in R satisfying a polynomial identity.

(6) There exists an essential left ideal in R with a polynomial identity.

Proof. The equivalence of the above statements follows directly.

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