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A REMARK ON PRIMITIVE RINGS AND J-PIVOTAL MONOMIALS

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1. For quite some time it has been a problem of great interest to study rings satisfying a polynomial identity. The latter concept was generalized by Drazin [1] to the notion of a pivotal monomial (PM). The main structure theorem for rings satisfying a PM is that a right primitive ring with a right PM is a full matrix ring over a division ring. Later Amitsur [2] generalized the concept of a PM ring to that of a J-pivotal monomial ring (JPM). This note concerns the structure of primitive rings having a right ideal I, $I \neq 0$, where I is a JPM ring. The main theorem is: Let R be a primitive ring. Then R has a non-zero socle iff for some right ideal $I \neq 0$, I is a right JPM ring.

2. Let R be a ring. We say that R is *(right) J-pivotal monomial (JPM) ring* of degree d if and only if each (right) primitive homomorphic image of R is isomorphic to a full ring of $n \times n$ matrices, $n \leqq d$, over some division ring. Notice that the above definition of a JPM ring is only an equivalent form of the original definition given by Amitsur [2]. For the purpose of this note the above definition is more desirable.

We shall require the following results concerning JPM and PM rings. (1) If U is a right ideal in a JPM ring of degree d then U is a JPM ring of degree $h \leqq d$, [2]; (2) a primitive PM ring of degree d is a full matrix ring over a division ring.

Theorem. Let R be a primitive ring. Then R has a non-zero socle if and only if there is a non-zero right ideal I of R which is a JPM ring.

Proof. Assume $I \neq 0$ a right ideal, I a JPM ring. Let I° denote the radical of I. Then I/I° is primitive, hence $I/I^\circ \cong D_n$, D_n the ring of $n \times n$ matrices over a division ring D. Since D_n has idempotents so does I/I° and thus so has I. Let $e \in I$ be a non-zero idempotent. Then eR is a right ideal of I, hence eR is a JPM ring. Hence we may assume that I has the form eR for some idempotent $e \neq 0$. We shall show I contains

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a minimal right ideal, proceeding by cases on n. Assume $n = 1$. If $f \in I$, $f \neq 0$, $f^2 = f$ then in I/I° we have $\bar{f} - \bar{e} = 0$. Thus $f - e \in I^\circ$ and $(I^\circ)^2 = 0$, so we obtain $(f-e)^2 = 0$. Since $ef = f = f^2$ we then have $e - fe = 0$. Hence $f(eR) = feR = eR = I$. Since $fI \subseteq fR \subseteq I$ we see that $fR = I$. Now if $J \subseteq I$ is a non-zero right ideal of R then like I, J contains an idempotent $f \neq 0$. By the above we have $I = fR \subseteq J \subseteq I$, so $J = I$. Hence I is minimal. Suppose now that $n > 1$. Since n is the maximal size of sets of orthogonal idempotents in D_n, then in I there are orthogonal idempotents $e_1, e_2, \ldots, e_n$. It's clear that $e_1, e_2, \ldots, e_n$ is also a maximal set of orthogonal idempotents for I. Suppose e_1R is not minimal. Let J be a non-zero right ideal of I properly contained in e_1R. J is a right ideal contained in I, hence J is a JPM ring. Thus $J/J^\circ \cong D'_k$, D' a division ring. If $k = 1$, then, as in the case of $n = 1$, J is minimal. Assume $k > 1$; then J contains at least two orthogonal idempotents f_1, f_2. For $i > 1$, $e_iJ \subseteq e_i e_1 R = 0$, hence e does not belong to J. Thus $f_j \neq e_i$ for $i > 1$, $j = 1, 2$. Let $a_i = f_i e_1$, $i = 1, 2$. Then $a_i^2 = f_i e_1 f_i e_1 = f_i^2 e_1 = f_i e_1 = a_i$. Also $a_i \neq 0$ since $a_i = 0$ implies $f_i J = 0$ which in turn implies $f_i^2 = 0$. Moreover, $a_i a_j = f_i e_1 f_j e_1 = f_i f_j e_1 = 0$ if $i \neq j$, and $a_j e_i = f_j e_1 e_i = 0$, $e_i a_j = e_i f_j e_1 = e_i e_1 f_j e_1 = 0$ for $i > 1$. Hence $a_1, a_2, e_2, \ldots, e_m$ is a set of $m + 1$ orthogonal idempotents for I which is impossible. This contradiction shows that R contains a minimal right ideal. Conversely, suppose $I = eR$ is a minimal right ideal, $e^2 = e \neq 0$. Then eRe is a division ring. Hence for all $x, y \in R$ there is an $r \in eRe$ so that $exe \cdot eye = eye \cdot exe \cdot r$. Letting $\pi(\lambda)$ be the monomial $\lambda_1\lambda_2$, λ_i indeterminates and letting $\alpha(\lambda) = \lambda_2\lambda_1$, we see that for $x_1, x_2 \in I$, $\pi(x) I \subseteq \alpha(x) I$. Hence I is a strongly right PM-ring and thus a JPM-ring.

As a consequence of the above theorem we have

Corollary 1. Let R be a primitive ring having at most finitely many orthogonal idempotents. Let $I \neq 0$ be a right ideal of R, I a JPM ring. Then R possesses a pivotal monomial.

Proof. By the above theorem R has a non-zero socle. Since R has only finitely many orthogonal idempotents it follows that R is a full ring of $n \times n$ matrices over a division ring. Hence R possesses the PM, $\pi(\lambda) = \lambda^m$ in a single variable.

As a second consequence we have

Corollary 2. Let R be a primitive ring having at most finitely many orthogonal idempotents and let $I \neq 0$ be a right ideal satisfying a polynomial identity. Then R satisfies a polynomial identity.

Proof. As in Corollary 1 we obtain R is isomorphic to D_n, D a division ring. Letting $I = eR$, $e^2 = e$ we can choose $D = eRe$; since $D \subseteq I$, D satisfies a polynomial identity and thus is finite-dimensional over its center. Thus D_n is finite-dimensional and so satisfies a polynomial identity.

The condition of R having at most finitely many orthogonal idempotents cannot be removed from the hypotheses of Corollary 1 or 2. For an example see [3].

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