POLYNOMIAL RINGS WITH A PIVOTAL MONOMIAL[1]

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1. Amitsur in his paper on Finite Dimensional Central Division Algebras [1] has proved that in a division ring D with center C, $(D:C)\leqq n^2<\infty$ if and only if every primitive homomorphic image of a polynomial ring $D[x]$ is a complete matrix ring A_h, $h\leqq n$, over a division ring A. Equivalently speaking, a division ring is finite dimensional over its center if and only if the polynomial ring over it has a *J-pivotal monomial* (written as JPM). The object of this note is to show that if R is a ring with a nilpotent (Jacobson) radical then the polynomial ring $R[x]$ has a JPM if and only if $R[x]$ has a polynomial identity. Amitsur's result then follows as a special case of our result. Our proof of Theorem 1, in obtaining sufficiency, is on the same lines as that of Amitsur.

2. We begin with

THEOREM 1. *Let R be a primitive algebra over its centroid C. Then $(R:C)\leqq n^2<\infty$ if and only if every primitive homomorphic image of $R[x]$ is a complete matrix ring A_h, $h\leqq n$, over a division ring A.*

PROOF OF THE THEOREM: NECESSITY. Let $(R:C)\leqq n^2<\infty$. Then it is well known that R satisfies a minimal polynomial identity $S_d(x)=\sum \pm x_{i_1}x_{i_2}\cdots x_{i_d}$, of degree $d\leqq 2n$. This identity also holds in $R[x]$. Since a primitive ring with a polynomial identity of degree d is a central simple algebra with a dimensionality $\leqq [d/2]^2$, it follows that each primitive homomorphic image of $R[x]$ is a central simple algebra of dimension $\leqq [d/2]^2$; and therefore it is isomorphic to A_r for some division algebra A and for $r\leqq d/2\leqq n$. This proves necessity.

Before we obtain sufficiency we recall for convenience the definition of a J-pivotal monomial in a ring. Let $\lambda_1,\cdots,\lambda_t$ be a set of noncommutative indeterminates and let $\pi(\lambda)=\lambda_{i_1}\cdots\lambda_{i_d}$ be a monomial of degree d in the λ_i. Let P_π denote the set of all monomials $\sigma(\lambda)=\lambda_{j_1}\cdots\lambda_{j_q}$ such that either $q>d$ or $q\leqq d$ with $j_h\neq i_h$ for some $h\leqq q$. We call a monomial $\pi(\lambda)$ a right J-pivotal monomial for a ring R if for every substitution $\lambda_i=x_i\in R$, $\pi(x)r$ is right-quasi-regular

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mod $\sum_{\sigma \in P_\pi} \sigma(x)R$, for all $r \in R$. A ring with a right J-pivotal monomial is called a right JPM-ring. Henceforth a JPM-ring shall mean a right JPM-ring. It is proved in [2] that a ring R has a J-pivotal monomial of degree d if and only if every (right) primitive homomorphic image of R is a full matrix ring D_h over a division ring D with $h \leqq d$. A simple but an important consequence of the definition of a JPM-ring may be recorded in

SUBLEMMA. *A homomorphic image of a JPM-ring is also a JPM-ring. In particular, if $R[x]$ has a JPM then its homomorphic image R is also a JPM-ring.*

Sufficiency. Let R be a primitive ring such that every primitive homomorphic image of $R[x]$ is a complete matrix ring A_h, $h \leqq n$, over a division ring A, viz., $R[x]$ has a JPM of degree n. So that by the sublemma R has JPM of degree n and consequently, it is full matrix ring A_h, $h \leqq n$ over a division ring A. Therefore we have

$$R[x] = A_h[x] \cong (A[x])_h.$$

We can assume that

$$R[x] = (A[x])_h = S_h, \qquad S = A[x].$$

Consider the maximal right ideal

$$I = (x-a)A[x], \qquad a \in A.$$

We note that each primitive ideal of $A[x]$ will be maximal ideal of $A[x]$. Therefore if $P = p(x)A[x]$ ($A[x]$ is a principal ideal ring) be a primitive ideal contained in I, then P is a maximal ideal in $A[x] = S$. Since S has unity, S/P is a simple primitive ring. Then the isomorphism

$$S_h/P_h \cong (S/P)_h$$

gives that S_h/P_h is a primitive ring. Accordingly, $S_h/P_h \cong D_r$ with $r \leqq n$. Further if $I_u = (x - uau^{-1})A[x]$, $0 \neq u \in A$, then it can be verified that

$$P_h = \cap (I_u)_h.$$

Since $S_h/P_h \cong D_r$, we can find r elements $u_1, \cdots, u_r$ such that

$$A_h[x] \supset (I_{u_1})_h \supset (I_{u_1})_h \cap (I_{u_2})_h \supset \cdots$$
$$\supset (I_{u_1})_h \cap (I_{u_2})_h \cap \cdots \cap (I_{u_r})_h = P_h.$$

Observing that $(I_u)_h = (x - uau^{-1})A_h[x]$, we can claim that $p(x)$ is a left common divisor of polynomials $x - u_i a u_i^{-1}$ and therefore degree of

$p(x) \leqq r$. It follows therefore that for each a in A there exists a polynomial $p(x)$ of degree $\leqq n$ with coefficients in center such that $x-a$ is a right divisor of $p(x)$. Hence $p(a)=0$. This implies A is an algebraic algebra of bounded degree. By Kaplansky [5] A satisfies a polynomial identity and is finite dimensional over its center. Hence $R=A_h$ is finite dimensional over its center (=centroid, since R has a unity). This completes the proof.

Next we prove

THEOREM 2. *Let R be a ring having its (Jacobson) radical nilpotent. Then $R[x]$ has JPM if and only if $R[x]$ has PI.*

PROOF: NECESSITY. Let J be radical of R and $J^m=0$. Let $R[x]$ have JPM of degree n. Let $\overline{P}$ be a primitive homomorphic image of $\overline{R}=R/J$. Then this, along with natural homomorphism induces the diagram

$$R[x] \rightarrow \overline{R}[x] \rightarrow \overline{P}[x].$$

By the sublemma $\overline{P}[x]$ has JPM and therefore Theorem 1 gives that $\overline{P}$ satisfies a standard identity of degree $\leqq 2n$. Consequently, $\overline{R}$ which is a subdirect sum of its primitive images satisfies a standard identity $S_d(x)=0$ of degree $d \leqq 2n$. This implies R satisfies $[S_{2n}(x)]^m=0$. The sufficiency is easy and therefore omitted.

REMARK 1. The theorem is still true for a ring R having its radical satisfying some polynomial identity. For if J satisfies an identity $p(x_1, \cdots, x_k)=0$, then R will satisfy $p[S_{2n}(x_1', \cdots, x_{2n}'), \cdots, S_{2n}(x_1^k, \cdots, x_{2n}^k)]=0$.

REMARK 2. The theorem is also true for a ring R with a strongly pivotal monomial and nil radical. For, in this case, radical will be nilpotent.

Belluce and Jain [3] have shown that a primitive ring satisfies a polynomial identity if and only if (1) it has at most a finite number of orthogonal idempotents (written as FI-ring), and (2) it has a nonzero one-sided ideal satisfying some polynomial identity. This result along with Theorem 2 gives the following,

THEOREM 3. *Let R be a primitive algebra over its centroid C. Then $(R:C) \leqq n^2 < \infty$ if and only if R is an FI-ring having a nonzero one-sided ideal I such that every primitive homomorphic image of $I[x]$ is a complete matrix ring A_h, $h \leqq n$, over a division ring A.*

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