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Rings Having One-Sided Ideals Satisfying a Polynomial Identity

By
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Introduction. The problem of determining the structure of a ring in which certain special subset satisfies a polynomial identity has recently found interest with some authors including Amitsur, Herstein, Martindale and Belluce [9, 16, 17 and 3]. It is shown by Belluce and Jain [3] that if R is a prime ring which possesses a non-zero right ideal A with a polynomial identity then R satisfies a polynomial identity if any of the following conditions hold: (1) $l(A) = 0$, (2) R is a right Goldie ring. The object of the present paper is to study rings, not necessarily prime, which possess a non-zero right ideal A satisfying a polynomial identity. In contrast to the prime case, examples are given to show that (i) a ring R may possess a two-sided ideal A with a polynomial identity and $l(A) = 0$ but the ring itself may not satisfy any polynomial identity and (ii) a Goldie ring may fail to possess a polynomial identity even though it possesses a two-sided ideal with a polynomial identity. Section 3 is devoted to sharpen some of the results proved earlier for prime rings [3, 4]. Sufficient conditions are obtained in sections 4 and 5 that the maximal quotient rings of semi-prime rings and artinian rings satisfy a polynomial identity, whenever they possess a non-zero right ideal A such that A satisfies a polynomial identity and $l(A) = 0$.

1. Preliminaries and Definitions. For a ring R the symbols $M^{\triangle}$, $R^{\triangle}$, $L^{s}(R)$, $L^{\triangle}(R)$ and $\hat{R}$ respectively will denote as usual the singular submodule of an R-module M, the right singular ideal, the lattice of all closed right ideals, the lattice of all large right ideals and the maximal (right) quotient ring in the sense of Johnson [13] and we denote by $l_S(X)$ the left annihilator of a subset X of R in a subset S of R. It is known that if R is a ring with $R^{\triangle} = 0$, then $\hat{R}$ can be looked upon as $\bigcup \mathrm{Hom}_R(A, R)$, where A is a large right ideal of R, and further $\hat{R}$ is a (Von-Neumann) regular ring which as a right R-module is the unique maximal essential extension of R as an R-module [11]. Thus by Eckmann and Schopf [5] $\hat{R}$ is also injective as a right R-module. It is also proved by Johnson and Wong ([14], theorem 7) that $\hat{R}$ is right self-injective. Therefore by Johnson ([12], p. 542) each closed right ideal of $\hat{R}$ is a direct summand of $\hat{R}$. But this implies each member A of $L^{s}(\hat{R})$ is also injective as a right R-module. Hence A is injective hull of $A \cap R$ [5]. The lattices of closed right ideals of $\hat{R}$ and R are known to be isomorphic by the mapping $A \rightarrow A \cap R$ ([12], theorem 6.8). The maximal quotient ring of a semi-prime Goldie ring is known to coincide with the classical quotient ring (cf. theorem 4.4, [13]).

We also recall the definition of a quasi-standard identity [4]. A ring R is said to satisfy a quasi-standard identity (QSI) of degree d if for each d-tuple $(r_1, \ldots, r_d)$ there exist a positive integer n such that

$$\Big(\sum_g \pm r_{g(1)} \cdots r_{g(d)}\Big)^n = 0\,,$$

where the summation runs over all the permutations g of $1, \ldots, d$ and the sign is positive or negative according as the permutation is even or odd. It was shown in [4] that a prime ring with a right singular ideal zero and uniform right ideals is a right Goldie ring if it has a quasi-standard identity. We shall obtain this result as a corollary to one of the theorems proved below.

Throughout this paper we assume that R is an algebra over a field F. If A is a non-zero right ideal satisfying some polynomial identity of degree d and $l_R(A) = 0$, then since AR is an algebra right ideal contained in A and $l_R(AR) = 0$, we can assume that A is an algebra right ideal with a polynomial identity and $l_R(A) = 0$.

2.1. Example (Amitsur). Let D be a division algebra infinite dimensional over its center C. Consider the ring R of all triangular matrices of the form $\begin{pmatrix} x & y \\ 0 & k \end{pmatrix}$, where x and y are in D and k is in C. R has the Jacobson radical

$$N = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in D\right\}, \quad N^2 = 0 \quad \text{and} \quad R/N \cong D \oplus C\,.$$

Therefore R/N is semi-simple artinian and it satisfies no polynomial identity since D cannot satisfy any polynomial identity. Consequently, R cannot satisfy any polynomial identity. But R has an ideal

$$A = \left\{\begin{pmatrix} 0 & y \\ 0 & k \end{pmatrix} : y \in D,\ k \in C\right\}$$

such that (i) $l_R(A) = 0$ and (ii) A satisfies the identity $(X_1X_2 - X_2X_1)^2 = 0$.

2.2. Example. Let D be an infinite dimensional algebra over its center C. Let F be any field. Then $D \oplus F$ is a Goldie ring having a two-sided ideal satisfying a polynomial identity. However $D \oplus F$ satisfies no polynomial identity.

3. Prime Rings. We give a relationship between the degrees of polynomial identities satisfied by the right ideal A in a prime ring R and the ring R in the theorem 1 of [3]. We will also need this result later on in sections 4 and 5.

3.1. Theorem. *Let R be a prime ring. If A is a non-zero right ideal satisfying a polynomial identity of degree d and $l_R(A) = 0$, then R satisfies a standard identity of degree d.*

Proof. Since A is a prime ring by itself, therefore, by Posner [18] the quotient ring of A exists and it satisfies same multilinear identities as satisfied by A. But A satisfies a multilinear identity of degree $d' \leqq d$. Because the quotient ring of A is semi-simple artinian, by Amitsur [1], the quotient ring satisfies a standard identity of degree d'. It is shown in the proof of theorem 1 in [3] that R is embeddable in the quotient ring of A. Hence R satisfies a standard identity of degree d'. But then R also satisfies a standard identity of degree d, because $d' \leqq d$.

We now prove that the maximal quotient ring satisfies a generalized polynomial identity (for definition see AMITSUR [2]).

3.2. Theorem. *Let R be a prime ring such that R has a zero right singular ideal and has uniform right ideals. If there exists a non-zero right ideal A in R such that A satisfies a quasi-standard identity then the maximal quotient ring of R satisfies a generalized polynomial identity.*

Proof. It is well known that under the given conditions on R, each right ideal contains a uniform right ideal. Let U be a uniform right ideal in A. Then U has QSI. Let f be a mapping of U to $\mathrm{Hom}_R(U, U)$, given by $a \to l_a$, where l_a denotes the left multiplication by a, f is then a non-zero homomorphism and $f(U)$ is a left ideal in $\mathrm{Hom}_R(U, U)$. Thus $K = \mathrm{Hom}_R(U, U)$ which is an integral domain has a left ideal with QSI. But this implies K has a left ideal with SI and hence by theorem 3.1 K has SI. Thus by AMITSUR $\hat{K}$ has SI. But by FAITH and UTUMI [cf. 6],

$$\hat{K} = \mathrm{Hom}_{\hat{R}}(\hat{U}, \hat{U}) = \mathrm{Hom}_{\hat{R}}(e\hat{R}, e\hat{R}) = e\hat{R}e\,.$$

This implies the minimal right ideal $e\hat{R}$ has a polynomial identity and hence $\hat{R}$ has a generalized polynomial identity.

We deduce a result which is proved in [4] by using weak transitivity of R.

Corollary. *If the ring R satisfies a quasi-standard identity then the maximal quotient ring satisfies a polynomial identity and hence is a finite dimensional central simple algebra.*

Proof. If I is a minimal right ideal in $\hat{R}$, then $U = I \cap R$ is a uniform right ideal of R such that $\hat{U} = I$. Following the proof in the theorem we can show that each minimal right ideal of $\hat{R}$ satisfies the same polynomial identity. Hence the soçle has a polynomial identity. Since the maximal quotient ring is also prime, it follows by theorem 3.1 that it also satisfies a polynomial identity. This proves the corollary.

4. Semi-prime Rings. Lemma 4.1. which follows is well known (cf. LEVY [15]).

4.1. Lemma. *If a semi-prime ring T has acc on annihilator ideals then the set M of annihilator (two sided) ideals contains only a finite number of maximal members whose intersection is zero.*

4.2. Theorem. *If R is a semi-prime ring which has acc on annihilator two sided ideals and if there exists a non-zero right ideal A satisfying a polynomial identity such that $l_R(A) = 0$, then the maximal quotient ring of R also satisfies a polynomial identity.*

Proof. Let B be any two sided ideal of R. If for any a in R, $Ba = 0$, then for any α in F, $B(a\alpha) = 0$. Therefore, annihilator ideal is an algebra ideal. By 4.1 there exist a finite number of distinct maximal annihilator ideals, say, $A_1, \ldots, A_n$ with zero intersection. Thus $R_i = R/A_i$ is a prime ring which is an algebra over F. If g_i is the natural homomorphism of R onto R_i, then it is easy to prove that $l_{R_i}(g_i(A)) = 0$.

Thus by theorem 3.1, R_i satisfies a standard identity of degree d. By POSNER [18], the classical quotient ring $Q(R_i)$ (which is same as $\hat{R}_i$ [13]) exists and satisfies the same identity. Following the lines of proof of theorem 4.7 in [8], we can show that $Q(R)$ exists and is isomorphic to $\oplus \sum Q(R_i)$. Consequently, $Q(R)$ also satisfies a standard identity. Now each $Q(R_i)$, we know, is simple artinian. Hence by [7], theorem 4.4, R is a semi-prime Goldie ring. But by JOHNSON [13], $\hat{R} = Q(R)$. This completes the proof.

A consequence of the above is the following result proved by SMALL [19].

Corollary. *If the ring R satisfies a polynomial identity, then the classical quotient ring $Q(R)$ also satisfies a polynomial identity.*

Our next theorem is concerned with a semi-prime ring having its socle as a large right ideal. The proof depends on the following lemma which is interesting by itself.

4.3. Lemma. *If T is any semi-prime ring such that its socle X is a large right ideal, then*

$$\hat{T} = \prod_i \mathrm{Hom}_{X_i}(X_i, X_i),$$

where X_i are the homogeneous components of the socle of T.

Proof. Since each right ideal of T contains an idempotent, $T^{\triangle} = 0$. Further the socle X is a large right ideal of T, therefore $X^{\triangle} = 0$ and hence the maximal quotient ring of X and that of T are same. But in a semi-prime ring a minimal right ideal is also a minimal right ideal of its socle (as a ring). Therefore the socle X of T is completely reducible as a right X-module. Consequently, each right ideal of X is a direct summand of X. Therefore if A is a large right ideal of X then $A = X$. But $\hat{X} = \bigcup \mathrm{Hom}_X(A, X)$ where A is a large right ideal of X. This gives $\hat{X} = \mathrm{Hom}_X(X, X)$. Let $X = \oplus \sum_i X_i$, where X_i are the homogeneous components of X. Therefore,

$$\mathrm{Hom}_X(X, X) \cong \prod_i \mathrm{Hom}_X(X_i, X) = \prod_i \mathrm{Hom}_X(X_i, X_i).$$

But X_i is a direct summand of X, therefore, $\mathrm{Hom}_X(X_i, X_i) = \mathrm{Hom}_{X_i}(X_i, X_i)$. Hence $\mathrm{Hom}_X(X, X) = \prod_i \mathrm{Hom}_{X_i}(X_i, X_i)$. This completes the proof.

Remark. It is worth noticing that in the above lemma, semi-primeness of T can be replaced by $T^{\triangle} = 0$ and each minimal right ideal of T is a minimal right ideal of X (as a ring).

4.4. Theorem. *Let R be a semi-prime ring such that its socle is a large right ideal. If there exists a non-zero right ideal A satisfying a polynomial identity of degree d and $l_R(A) = 0$, then the maximal quotient ring of R satisfies a standard identity of degree d.*

Proof. Let A be the given right ideal with polynomial identity of degree d such that $l_R(A) = 0$. Let $A_i = A \cap S_i$, where S_i are the homogeneous components of S. Let x_i be in S_i such that $x_i A_i = 0$. This implies $x_i\left(\sum_j A_j\right) = 0$. Therefore $x_i A = 0$, because trivially $\sum_j A_j \subset' A$ i.e. A is an essential extension of $\sum_j A_j$ as R-modules and further $R^\Delta = 0$. But then $x_i = 0$. Hence $l_{S_i}(A_i) = 0$. Since S_i is a simple ring, by theorem 3.1, S_i satisfies a standard identity of degree d. Hence S_i is a full matrix ring $D^{(i)}_{n_i}$ over a division ring $D^{(i)}$. This implies $\mathrm{Hom}_{S_i}(S_i, S_i)$ is isomorphic to S_i and thus satisfies a standard identity of degree d. If we apply lemma 2 to R, we obtain $\hat{R} = \prod_i \mathrm{Hom}_{S_i}(S_i, S_i)$. Consequently $\hat{R}$ satisfies a standard identity of degree d.

The following simple example shows that the hypothesis in the above theorem is sufficient but not necessary.

4.5. Example. Let Z be the ring of integers and F be a field. Then $R = F \oplus \mathrm{Z}$ is a semi-prime ring with a polynomial identity but the socle is not large.

5. Artinian Rings. We assume now that R is a right artinian ring with zero right singular ideal. Let S denote the (right) socle of R, which is a large right ideal in R.

5.1. Lemma. *Any minimal right ideal I of R is a minimal right ideal of S and conversely.*

Proof. Let I be any minimal right ideal of R. Let $0 \neq x \in I$. Then $xS \neq 0$, since $R^\Delta = 0$. Therefore xS is a non-zero right ideal of R contained in I. Thus $xS = I$ and this gives that I is a minimal right ideal of S. Conversely let J be a minimal right ideal of S. Then again $JS \neq 0$, $JS \subset J$ and therefore $JS = J$. But JS is a right ideal of R. Therefore J is also a minimal right ideal of R.

5.2. Lemma. *Any minimal right ideal of R contained in a homogeneous component S_i of S is a minimal right ideal of S_i and conversely.*

Proof. It follows from lemma 5.1 and the fact that S_i is a direct summand of S.

5.3. Lemma. $\hat{S} = \hat{R} = \mathrm{Hom}_S(S, S) = \oplus \sum \hat{S}_i$, *where S_i are homogeneous components of S.*

The proof follows from the 5.2 and the remark following 4.3.

We are now in a position to prove one of the main results of this paper.

5.4. Theorem. *Let R be a right artinian ring such that it has a zero right singular ideal. If there exist a non-zero right ideal A satisfying a polynomial identity and $l_R(A) = 0$ then the maximal quotient ring of R satisfies a polynomial identity and is therefore a finite direct sum of finite dimensional central simple algebras.*

Proof. Let $A_i = A \cap S_i$ where A is the given right ideal with polynomial identity. It follows on the same lines as in the proof of theorem 4.4 that $l_{S_i}(A_i) = 0$. Now S_i is a finite direct sum of mutually isomorphic right ideals of R (and therefore of S_i because of lemma 5.2). Let $S_i = \oplus \sum_j A_{ij}$, where A_{ij} are minimal right ideals of S_i which are isomorphic to each other. Each A_{ij} is a direct summand and therefore it is a closed right ideal of S_i. In the lattice $L^s(S_i)$ of closed right ideals of S_i, we have $S_i = \bigvee_j A_{ij}$ an irredundant decomposition into atoms of $L^s(S_i)$. Let B_{ij} be the members of $L^s(\hat{S}_i)$ corresponding to A_{ij} in the isomorphism between the lattices

$L^s(S_i)$ and $L^s(\hat{S}_i)$. Then we get $\hat{S}_i = \bigvee_j B_{ij}$ an irredundant decomposition of $\hat{S}_i$ into atoms of $L^s(\hat{S}_i)$. This gives $\hat{S}_i = \oplus \sum_j B_{ij}$. As explained in section 2, each B_{ij} is injective hull of A_{ij} as R-module. Thus B_{ij} are mutually isomorphic as R-modules. But $R^\Delta = 0$ implies that they are mutually isomorphic as $\hat{R}$-module. Hence $\hat{S}_i$ is a direct sum of finite number of mutually isomorphic minimal right ideals of $\hat{S}_i$. Since $\hat{S}_i$ is also regular it is a full matrix ring over a division ring

$$D^{(i)} = \mathrm{Hom}_{\hat{S}_i}(B_{ij}, B_{ij})$$

where B_{ij} is a minimal right ideal of $\hat{S}_i$.

But if $f \in \mathrm{Hom}_{S_i}(B_{ij}, B_{ij})$, then restricting f to the irreducible S_i-module A_{ij}, we have $f(A_{ij}) = 0$ or $f(A_{ij})$ is isomorphic to A_{ij}. Since B_{ij} is also uniform as an S_i-module, $A_{ij} \cap f(A_{ij}) \neq 0$, when $f(A_{ij}) \neq 0$. Thus $A_{ij} = f(A_{ij})$. Further if f is in $\mathrm{Hom}_{S_i}(A_{ij}, A_{ij})$ then f can be uniquely extended to $\mathrm{Hom}_{S_i}(B_{ij}, B_{ij})$, because B_{ij} is injective hull of A_{ij} as S_i-module and trivially singular submodule of A_{ij} is zero. Hence $\mathrm{Hom}_{S_i}(A_{ij}, A_{ij}) = \mathrm{Hom}_{S_i}(B_{ij}, B_{ij})$. However,

$$\mathrm{Hom}_{S_i}(B_{ij}, B_{ij}) = \mathrm{Hom}_{\hat{S}_i}(B_{ij}, B_{ij}),$$

since $S_i^\Delta = 0$. Therefore we obtain $\mathrm{Hom}_{S_i}(A_{ij}, A_{ij}) = \mathrm{Hom}_{\hat{S}_i}(B_{ij}, B_{ij}) = D^{(i)}$.

Let N_i be the radical of S_i. N_i is nilpotent. Thus $A_i \not\subseteq N_i$, as $l_{S_i}(A_i) = 0$. Hence there exists a minimal right ideal I_i of S_i contained in A_i such that $I_i \cap N_i = 0$. Then $I_i = e_i S_i$ for some idempotent e_i in S_i and we have

$$D^{(i)} = \mathrm{Hom}_{S_i}(A_{ij}, A_{ij}) = \mathrm{Hom}_{S_i}(I_i, I_i) = e_i S_i e_i.$$

Now $e_i S_i e_i \subseteqq A_i \subseteqq A$. Therefore $e_i S_i e_i$ satisfies same PI as satisfied by A. Consequently $D^{(i)}$ also satisfies a PI. Then KAPLANSKY's theorem states that $D^{(i)}$ is finite dimensional over its center. Then $\hat{S}_i = D^{(i)}_{n_i}$ is also finite dimensional over its center and $\hat{S}_i$ satisfies some standard identity of degree say m_i. If we set

$$m = \max(m_1, m_2, \ldots, m_k),$$

then each $\hat{S}_i$ satisfies the standard identity of degree m. Now by lemma 5, $\hat{R} = \oplus \sum \hat{S}_i$. Hence $\hat{R}$ also satisfies the standard identity of degree m.

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