GENERALIZED COMMUTATIVE RINGS

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Dedicated to the memory of TADASI NAKAYAMA

Among his various interests in algebra Nakayama also took part in the various researches, published in the early and middle 1950's, which dealt with the commutativity of rings [4, 5]. This paper, which studies a problem of a related sort, thus seems appropriate in a Journal honoring his memory.

We shall study a certain class of rings which satisfy a weak form of the commutative law and shall show that the structure of such rings can be determined.

DEFINITION. A ring R is said to be a *generalized commutative* ring (written as g.c. ring) if given $x, y \in R$ there exists positive integers $m(x, y)$, $n(x, y)$ depending on x and y such that $(xy)^{n(x,y)} = (yx)^{m(x,y)}$.

We begin the study of g.c. rings with

THEOREM 1. *Let D be a division ring which is a g.c. ring; then D is commutative.*

Proof. Let $a, b \in D$, $a \neq 0$, $b \neq 0$; using $x = a$, $y = ba^{-1}$ in the definition of g.c. ring we know that there are positive integers n and m such that

(1) $$ab^n a^{-1} = (aba^{-1})^n = (a(ba^{-1}))^n = ((ba^{-1})a)^m = b^m.$$

Therefore $ab^{n^2}a^{-1} = b^{mn}$ whence $a^2b^{n^2}a^{-2} = ab^{mn}a^{-1} = (ab^n a^{-1})^m = b^{m^2}$. Continuing, we easily obtain

(2) $$a^i b^{n^i} a^{-i} = b^{m^i} \qquad \text{for all } i > 0.$$

Since D is a g.c. ring there exist positive integers k and t such that

(3) $$b^{-n}a^k b^n = (b^{-n}ab^n)^k = a^t.$$

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Put $i=k$ in (2) and conjugate the result by b^n. We get

$$(4) \qquad a^t b^{n^k} a^{-t} = b^{m^k}.$$

Since, however, $a^k b^{n^k} a^{-k} = b^{m^k}$ we see that a^{t-k} commutes with b^{n^k}. If $k=t$ then (3) tells us that a positive power of a, namely a^k, commutes with a positive power of b, namely b^n. If $k \neq t$ the remark made above shows us that a positive power of a, namely $a^{|k-t|}$, commutes with the positive power b^{n^k} of b. We have thus shown that for any $a, b \in D$ there exists positive integers $r(a, b)$, $s(a, b)$ such that $a^{r(a,b)}$ commutes with $b^{s(a,b)}$. The proof of Theorem 1 will therefore be complete when we have established

LEMMA 1. *Let D be a division ring in which given any a, b there exist positive integers $r(a, b)$, $s(a, b)$ such that $a^{r(a,b)}$ commutes with $b^{s(a,b)}$. Then D is commutative.*

Proof. If for every x, y in D, x commutes with some positive power of y, then by a result of Herstein [3] D would be commutative.

Suppose then that we can find $a, b \in D$ such that a commutes with *no* b^n for $n>0$. Let $W = \{x \in D \mid xb^{m(x)} = b^{m(x)}x \text{ for some } m(x)>0\}$; Clearly W is *a* subdivision ring of D. Moreover, since $a \notin W$, $W \neq D$. By hypothesis, given $x \in D$, $x^{r(x,b)} \in W$. thus in Faith's terminology D is radical over the proper subdivision ring W. A theorem of Faith [1] then tells us that D is commutative.

Lemma 1 has some independent interest for from it one easily deduces that if R is *a* semi-simple ring in which $x^{r(x,y)}$ commutes with $y^{s(x,y)}$ for all x, y in R then R is commutative. This answers a question raised by Faith [2].

Since subrings and homomorphic images of g.c. rings are g.c. rings and since the $n \times n$ matrices over a division ring (or any other ring with unit, for that matter) are not g.c. rings if $n>1$ we easily pass from the division ring case to the primitive case and thereby to the semi-simple case to obtain

THEOREM 2. *A semi-simple g.c. ring is commutative.*

COROLLARY. *If R is a g.c. ring, $J(R)$ its Jacobson radical and $C(R)$ its commutator ideal then $C(R) \subset J(R)$.*

We now are ready to prove the main result of this note

THEOREM 3. *The commutator ideal of a g.c. ring is nil.*

Proof. Let $J(R)$ be the Jacobson radical of R and $C(R)$ the commutator ideal of R. As we have just seen, $C(R) \subset J(R)$. Thus we may suppose that $J(R) \neq (0)$. By factoring out the maximal nil ideal of R we may suppose that R has no non-zero nil ideals. Our objective then becomes to prove that R is commutative, that is, $C(R) = (0)$. Suppose $C(R) \neq (0)$.

We first claim that $J(R)$ can not be commutative, for if it were for $a, b \in J(R)$ and $y \in R$ then since $ay \in J(R)$, $aby = b(ay) = (ay)b$, hence $a(by - yb) = 0$. Since $by - yb$ is in $J(R) \cap \{x \in R \mid J(R)x = (0)\}$ which is a nilpotent ideal, we get that $by - yb = 0$ for all $b \in J(R)$, $y \in R$, that is $J(R) \subset Z(R)$, the center of R. Given $a \in J(R), x, y \in R, ax \in J(R) \subset Z(R)$ hence $(ax)y = y(ax) = yax = ayx$, leading to $a(xy - yx) = 0$. Since $J(R) \subset Z(R)$ this immediately implies that $J(R)C(R) = (0)$; together with $C(R) \subset J(R)$ we obtain $C(R)^2 = (0)$. Since R has no non-zero nilpotent ideals this latter forces $C(R) = (0)$, contrary to assumption.

We may therefore assume that $J(R)$ is not commutative. Since R has no non-zero nil ideals then $J(R)$ as a ring in its own right also has no non-zero nil ideals. Since $J(R)$ has no nil ideals by the previously cited theorem of Herstein there are two elements $a, b \in J(R)$ such that a commutes with *no* b^n for $n > 0$. Clearly then, *b can not be nilpotent.*

Since $a \in J(R)$, $1 - a$ is formally invertible (R need not have a unit element) and the mapping $x \to (1-a)x(1-a)^{-1}$ is an automorphism of R.

In the hypothesis that R is a g.c. ring let $x = (1-a)b$, $y = b(1-a)^{-1}$. Thus there are positive integers r, s such that

$$((1-a)b^2(1-a)^{-1})^r = (b(1-a)^{-1}(1-a)b)^s = b^{2s},$$

that is, there are positive integers $m, n > 0$ ($m = 2r$, $n = 2s$) with

$$(1-a)b^m = b^n(1-a). \tag{1}$$

Since a commutes with no positive power of b the integers m, n in (1) satisfy $m \neq n$. We may suppose that $m > n$, otherwise we could carry out the argument on a' instead of a where $(1-a)^{-1} = 1 - a'$, that is, $a + a' - aa' = 0$.

Since both ab and ba are in $J(R)$ the same reasoning yields

$$(1-ab)b^m = b^p(1-ab) \tag{2}$$

$$(1-ba)b^m = b^q(1-ba) \tag{3}$$

(We can suppose that the powers of b on the left hand side of (1), (2) and (3) are the same for if $(1-a)b^{m_1}(1-a)^{-1}=b^{n_1}$, $(1-ab)b^{m_2}(1-ab)^{-1}=b^{n_2}$, $(1-ba)b^{m_3}(1-ba)^{-1}=b^{n_3}$ then $m=m_1m_2m_3$ satisfies the conditions of (1), (2) and (3)).

Multiply (2) from the left by b and (3) from the right by b; this yields

$$(b-bab)b^m=b^p(b-bab)$$
$$(b-bab)b^m=b^q(b-bab),$$

hence $(b^p-b^q)(b-bab)=0$.

If $p \neq q$ since b is in the radical of R (and so $(1-b^k)x=0$ forces $x=0$ when $k \geq 1$) we get $b^{r+1}(1-ab)=0$ where $r=\min(p, q)$. Since $1-ab$ is invertible we are left with $b^{r+1}=0$, contrary to the fact that b is not nilpotent. Therefore we conclude that $p=q$; (2) and (3) then become

$$(1-ab)b^m=b^p(1-ab)$$
$$(1-ba)b^m=b^p(1-ba).$$

Subtracting we get

$$(ab-ba)b^m=b^p(ab-ba). \tag{4}$$

We return to the interrelation of (1) and (2). Multiply (1) from the right by b and subtract from (2). We get

$$b^{m+1}-b^m=b^{n+1}-b^p+(b^n-b^p)ab. \tag{5}$$

If $b^n=b^p$ since $b \in J(R)$ we get $b^k=0$, where $k=\min(n, p)$, if $n \neq p$ contrary to b not nilpotent, hence $b^n=b^p$ forces $n=p$. But then (5) reduces to $b^{m+1}-b^m=b^{n+1}-b^n$. Since $m>n$, $b^n(1-b-b^{m-n}+b^{m+1-n})=0$; however $b+b^{m-n}-b^{m+1-n}$ being in $J(R)$, $1-b-b^{m-n}+b^{m+1-n}$ is invertible. The net result of this would be $b^n=0$, a contradiction. We thus know that $b^n \neq b^p$.

We commute (5) with b to get $(b^n-b^p)(ab-ba)b=0$. Since $n \neq p$ and $b \subset J(R)$ this yields

$$b^t(ab-ba)b=0 \qquad \text{where } t=\min(n, p). \tag{6}$$

From (4) we know that

$$(ab-ba)b^{km}=b^{kp}(ab-ba) \qquad \text{for any } k>0$$

so, multiplying (6) from the left by b^{kp-t} where $kp<t$ we get $(ab-ba)b^{km+1}=0$, so certainly $(ab-ba)b^{(k+1)m}=0$. Since $0=(ab-ba)b^{(k+1)m}=b^{(k+1)p}(ab-ba)$, if

$w = \max((k+1)n, (k+1)p)$ then $(ab-ba)b^w = b^w(ab-ba) = 0$. A simple induction on i reveals that

$$0 = (ab^i - b^i a)b^w = b^w(ab^i - b^i a) \qquad \text{for all } i > 0.$$

Put $i = w$; we then have $(ab^w - b^w a)b^w = 0$ and $b^w(ab^w - b^w a) = 0$. The net result of these is that $ab^{2w} = b^w ab^w = b^{2w}a$, that is, a commutes with a positive power of b. This we know is contrary to assumption. The theorem is thereby proved.

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