

SEMIPRIME RINGS WITH FINITE LENGTH W.R.T. AN IDEMPOTENT KERNEL FUNCTOR

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ABSTRACT

The purpose of this note is to show that the only idempotent kernel functor σ on $\text{Mod-}R$, where R is a semiprime ring and R_R has finite σ -length, is the idempotent kernel functor Z corresponding to the Goldie torsion theory.

The concept of σ -chains and σ -length of a module was introduced by Goldman [2]. For all our terminology and notations we refer to Goldman [1] and [2].

Throughout all modules are right and unital. For any R -module M we denote by \hat{M} the injective hull of M and for any two R -modules $A, B, A \subset B$ denotes that A is an essential submodule of B . Also for any idempotent kernel functor σ the set of right ideals A of the ring R such that $\sigma(R/A) = R/A$ is denoted by \mathcal{F}_σ and is called the topology with respect to σ . For any R -submodule N of M if $\sigma(M/N) = (0)$ then we say that N is a σ -closed submodule of M . The functor Z on the category $\text{Mod-}R$ is defined by $Z(M) = \{x \in M \mid xE = 0, \text{ for some essential right ideal } E \text{ in } R\}$, and it is, in general, a kernel functor. In case $Z(R) = (0)$ then Z is an idempotent kernel functor.

THEOREM. *Let R be a semiprime ring and σ be an idempotent kernel functor on $\text{Mod-}R$. Then the following are equivalent:*

- (i) R_R has finite σ -length.
- (ii) $\sigma(R) = (0)$, R is a right Goldie ring, and the classical quotient ring Q coincides with the ring of quotients $Q_\sigma(R)$.

In each case $\sigma = Z$ where Z is the idempotent kernel functor corresponding to the Goldie torsion theory.

PROOF. (i) \Rightarrow (ii). Since R_R has finite σ -length it follows that $\sigma(R) = (0)$ ([2], prop. 1.1). Thus right ideals in the topology corresponding to σ are all essentials, showing $\sigma \cong Z$. Further, R satisfies acc and dcc on σ -closed right ideals ([2], prop. 1.2). But it is obvious that every annihilator right ideal of R is σ -closed. Thus R satisfies acc and dcc on annihilator right ideals and so $Z(R) = (0)$. This implies R has finite Z -length ([2], cor. 1.3), proving that R is a right Goldie ring. Now R has a classical right quotient ring Q which is semisimple artinian, and $Q = \hat{R}_R$. As $\sigma(R) = (0)$, the quotient ring of R with respect to σ , namely, $Q_\sigma(R)$, is contained in Q . We proceed to show that $Q_\sigma(R) = Q$. Write $Q = \bigoplus_{i=1}^n e_i Q$, $e_i Q$ minimal right ideals of Q . Let $P_i = e_i Q \cap Q_\sigma(R)$. Since P_i has no essential extension in $Q_\sigma(R)$, each P_i is Z -closed and hence a σ -closed R -submodule of $Q_\sigma(R)$. Also it is easy to see that P_i is a minimal Z -closed R -submodule of $Q_\sigma(R)$. As $Q_\sigma(R)$ is σ -injective and P_i is a σ -closed R -submodule of $Q_\sigma(R)$, it follows that P_i is σ -injective. Further, $Q_\sigma(R)$ is a semiprime Goldie ring since $R \subset Q_\sigma(R) \subset Q$.

Now we show that P_i is a minimal right ideal of $Q_\sigma(R)$. Let A be a non-zero right ideal such that $A \subsetneq P_i$. Then $AP_i \neq (0)$. Let $a \in A$ such that $aP_i \neq (0)$. Since P_i is a minimal Z -closed R -submodule of $Q_\sigma(R)$, we get $P_i \cong aP_i \subsetneq P_i$. This gives an infinite properly descending chain $P_i \supsetneq a_1 P_i \supsetneq a_1 a_2 P_i \supsetneq \cdots$ of σ -injective right R -submodules and hence of σ -closed right R -submodules of $Q_\sigma(R)$ for some elements a_1, a_2, \dots , in A . However, by Goldman ([2], cor. 1.6 and prop. 1.2), $Q_\sigma(R)$ also satisfies acc and dcc on σ -closed right R -submodules. So the above properly descending chain leads to a contradiction. Hence each P_i is a minimal right ideal in $Q_\sigma(R)$. Clearly, $P = \bigoplus_{i=1}^n P_i \subset Q_\sigma(R)$. Thus $\text{socle}(Q_\sigma(R)) = P$. This yields $Q_\sigma(R) = \text{socle}(Q_\sigma(R))$ and hence $Q_\sigma(R) = Q$.

(ii) \Rightarrow (i). Obvious.

We now show that under any of the equivalent conditions in the theorem, $\sigma = Z$. We only need to show $Z \cong \sigma$ or equivalently $\mathcal{F}_Z \subset \mathcal{F}_\sigma$. Let $I \in \mathcal{F}_Z$ then $I \subset R$ and hence $\hat{I} = \hat{R} = Q$. Recall that $Q_\sigma(I) = \{x \in \hat{I} \mid xA \subset I \text{ for some } A \in \mathcal{F}_\sigma\}$. We assert that $Q_\sigma(I)$ is a right Q -module. Let $x \in Q_\sigma(I)$ and $q \in Q$. $x \in Q_\sigma(I)$ implies $xA \subset I$ for some $A \in \mathcal{F}_\sigma$. Also $A \in \mathcal{F}_\sigma$ gives $Q_\sigma(A) = Q_\sigma(R)$. Since $q \in Q = Q_\sigma(R) = Q_\sigma(A)$, there exists $B \in \mathcal{F}_\sigma$ such that $qB \subset A$. Now $xqB \subset xA \subset I$, yielding $xq \in Q_\sigma(I)$ and hence $Q_\sigma(I)$ is a right Q -module as asserted. But then $Q_\sigma(I) \subset Q$ implies $Q_\sigma(I) = Q$. Hence R/I is a submodule of $Q_\sigma(I)/I$. Therefore R/I is σ -torsion, proving that $I \in \mathcal{F}_\sigma$ as desired.

REMARK. Let R be a semiprime right Goldie ring with $\sigma(R) = 0$. Then $\sigma = Z$ if and only if R has finite σ -length.

REFERENCES

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