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# NOTE ON GENERALIZED COMMUTATIVE RINGS

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BELLUCE-HERSTEIN-JAIN have defined [1] a ring $R$ to be a generalized commutative ring (written as g.c. ring) if given $a, b \in R$ there exist positive integers $m = m(a, b)$, $n = n(a, b)$ such that $(ab)^m = (ba)^n$. A multiplicative semi-group $S$ will be said to have $H$-property if given $a, b \in S$ there exists a positive integer $n = n(a, b)$ such that $a^n b = ba^n$. In this note we provide an alternative proof to prove that the commutator ideal of a g. c. ring is nil. The lemma which we prove below has an independent interest also. It follows from the lemma that if $G$ is a multiplicative group in which for each $a, b \in G$, $(ab)^{m(a,b)} = (ba)^{n(a,b)}$ where $m(a, b)$ and $n(a, b)$ are positive integers then $G$ has $H$-property. We assume for convenience that the ring $R$ has unity.

LEMMA. *Let $\mathcal{G}$ be a multiplicative group. Let for $a, b \in \mathcal{G}$ there exist positive integers $m, n, r$ and $s$ depending on $a$ and $b$ such that $ab^m a^{-1} = b^n$ and $ba^r b^{-1} = a^s$. Then there exists a positive integer $\lambda$ such that $ab^\lambda = b^\lambda a$.*

PROOF. If $b$ is of finite order then the result is obvious. So let $b$ be not of finite order. Then if $b^m = b^n$, for positive integers $m$ and $n$, we must have $m = n$. We have by hypothesis $ab^m a^{-1} = b^n$. By induction we get $a^r b^{m^r} a^{-r} = b^{n^r}$ for all positive integers $r$. We write for convenience

$$a^r b^m a^{-r} = b^n. \qquad (1)$$

Consider the collection of all ordered pairs $(m_i, n_i)$ satisfying (1). Let $x(r)$ and $y(r)$ be the smallest positive integers among $m_i$'s and $n_i$'s respectively. We claim $a^r b^{x(r)} a^{-r} = b^{y(r)}$. For, let

$$a^r b^{x(r)} a^{-r} = b^y \tag{2}$$

$$a^r b^x a^{-r} = b^{y(r)}. \tag{3}$$

Raise (2) both sides by $x$ and (3) by $x(r)$. This would make left hand sides equal. Thus the right hand sides $b^{yx}$ and $b^{y(r)x(r)}$ are also equal. This by our remark in the beginning implies $yx = y(r).\,x(r)$. Since $x(r) \leqslant x$ and $y(r) \leqslant y$, we must have $x(r) = x$ and $y(r) = y$. Therefore, we have

$$a^r b^{x(r)} a^{-r} = b^{y(x)} \text{ for each positive integer } r. \tag{4}$$

Suppose we have also

$$a^r b^\lambda a^{-r} = b^\mu. \tag{5}$$

Then (4) and (5) yield $b^{\lambda y(r)} = b^{\mu x(r)}$. This means $\lambda y(r) = \mu x(r)$. So we obtain that for a given positive integer $r$, if the relation (5) is true, then the ratio $\dfrac{\lambda}{\mu}$ is constant and equals $\dfrac{x(r)}{y(r)}$.

Now, if $s$ is another positive integer,

$$a^{r+s} b^{x(r)x(s)} a^{-r-s} = (a^{r+s} b^{x(r)} a^{-r-s})^{x(s)}$$
$$= (a^s.a^r.b^{x(r)}.a^{-r}.a^{-s})^{x(s)} = (a^s.b^{y(r)}.a^{-s})^{x(s)}$$
$$= (a^s.b^{x(s)}.a^{-s})^{y(r)} = b^{y(s)y(r)}.$$

Therefore, by the remark just made before,

$$\frac{x(r)x(s)}{y(r)y(s)} = \frac{x(r+s)}{y(r+s)}.$$

If we set $f(r) = \dfrac{x(r)}{y(r)}$, then $f(r+s) = f(r)f(s)$.

This gives $f(r) = [f(1)]^r$. So that if we can prove $f(r) = 1$, for some $r$, then $f(r) = 1$ for each $r$.

In particular we would have for $r = 1$, the relation $ab^{x(1)} = b^{x(1)}a$. So we now proceed to show $f(r) = 1$ for some $r$. So far we have not used our second hypothesis

$$b.a^r b^{-1} = a^s. \tag{6}$$

(Note that $r$ and $s$ are some fixed positive integers satisfying this relation.) We conjugate (4) by $b$ and rewrite it as $ba^r.b^{-1}b^{x(r)}b.a^{-r}.b^{-1} = b^{y(r)}$. We use (6) to obtain $a^s b^{x(r)} a^{-s} = b^{y(r)}$. Also $a^r b^{x(r)} a^{-r} = b^{y(r)}$. Hence we obtain an integer $\lambda$ such that $a^\lambda b^{x(r)} a^{-\lambda} = b^{x(r)}$. Raise this both sides by $x(\lambda)$, we get $a^\lambda b^{x(r)x(\lambda)} a^{-\lambda} = b^{x(r)x(\lambda)}$. The left hand side is $b^{y(\lambda)x(r)}$. Thus $b^{y(\lambda)x(r)} = b^{x(r)x(\lambda)}$. But this implies $x(\lambda) = y(\lambda)$. Hence $f(\lambda) = 1$. This completes the proof.

THEOREM. *Let $R$ be a g.c. ring. Let $a$, $b \in R$. If $a$ and each $b^k$, where $k$ is a positive integer, are quasi-regular then there exists a positive integer $n = n(a, b)$, such that $ab^n = b^n a$.*

PROOF. If $b$ is nilpotent then the result is obvious. So let $b$ be not nilpotent. Then if $b^m = b^n$, for positive integers $m$ and $n$, we must have $m = n$. For, otherwise, let $m > n$. Then $b^n(1 - b^{m-n}) = 0$. Since $b^{m-n}$ is q.r., we get $b^n = 0$, a contradiction. Let $x = b(1-a)^{-1}$, $y = (1-a)b$. By hypothesis we have (after a little simplification) integers $m$, $n$, such that $(1-a)b^m(1-a)^{-1} = b^n$.

This is same as (1) in the lemma, with $(1-a)$ in place of $a$. Since $(1-a)$ has an inverse the argument in the lemma yields

$$(1-a)^r b^{x(r)} (1-a)^{-r} = b^{y(r)} \tag{A}$$

and we want to prove $x(r) = y(r)$ for some $r$. Again by hypothesis we have integers $r$ and $s$, such that

$$(1-b)(1-a)^r(1-b)^{-1} = (1-a)^s. \tag{B}$$

Multiply the equation (A) on the right by $(1-b)^{-1}$ and on the left by $(1-b)$. Then we get

$$(1-b)(1-a)^r(1-b)^{-1}b^{x(r)}(1-b)(1-a)^{-r}(1-b)^{-1} = b^{y(r)}.$$

Applying the equation (B), we obtain $(1-a)^s b^{x(r)}(1-a)^{-s} = b^{y(r)}$.

But this yields as in the lemma that there exists a positive integer $\lambda$, such that $(1-a)b^{x(\lambda)} = b^{x(\lambda)}(1-a)$.

Hence $(1-a)b^{x(1)} = b^{x(1)}(1-a)$, which gives $ab^{x(1)} = b^{x(1)}a$.

This completes the proof.

COROLLARY 1. *If $R$ is g.c. division ring then $R$ has the $H$-property.*

PROOF. If $a$ or some power of $b$ is identity then trivially there exists a positive integer $n$ such that $ab^n = b^n a$. In case neither $a$ nor any power of $b$ is identity then both $a$ and each power of $b$ is quasi-regular. Thus the theorem would give the result.

COROLLARY 2. *If $R$ is g.c. division ring then $R$ is a field.*
Follows from Corollary 1 and Herstein [2].

COROLLARY 3. *A semi-simple g.c. ring is commutative.*
The proof is usual deduction from the division ring.

COROLLARY 4. *If $R$ is a non-semi-simple g.c. ring then the Jacobson radical $J(R)$ has the $H$-property.*

Proof follows from the theorem.

COROLLARY 5. *If $R$ is a g.c. ring having no non-zero nil ideals, then $R$ is commutative.*

PROOF. $J(R)$ as a ring in its own right also has no non-zero nil ideals. Thus by Corollary 4 and Herstein [2], $J(R)$ is commutative. Since $R/J(R)$ is also a g.c. ring and is semi-simple, it is commutative by Corollary 3. Let $a, b \in J(R)$ and $x, y \in R$. Then $(ax)(by) = (by)(ax)$. This yields $(b(ax))y = (a(by))x$, so that $ab(xy - yx) = 0$. This means $J^2(R).C(R) = 0$, where $C(R)$ is a commutator ideal. Since $R/J(R)$ is commutative, $C(R) \subset J(R)$. Thus we get $C^3(R) = 0$. Hence $C(R) = 0$. So $R$ is commutative.

COROLLARY 6. *If $R$ is a g.c. ring then the commutator ideal of $R$ is nil.*

The proof is now obvious.

REMARK. We point out that the proof of the main result in [1], namely, the commutator ideal of a g.c. ring is nil, can also be shortened. The Theorem 3 therein proves that in a g.c. ring if $1 - ab$, $1 - ba$, and $1 - a$ have inverses then there exists a positive integer $n$ such that $a^n - b = ba^n$. This shows then a g.c. division ring has $H$-property and hence the theorem 1 in [1] does not need a separate argument.

## REFERENCES

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